HERMITIAN OPERATORS

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A hermitian operator T satisfies $T = T^{\dagger}$. [Some books refer to a hermitian operator as *self-adjoint* and some use the notation T^* for $T^{\dagger}$. Some books (such as Axler) also denote a complex conjugate by a bar over a symbol rather than an asterisk.]

As preparation for discussing hermitian operators, we need the following theorem.

Theorem 1. *If T is a linear operator in a complex vector space V, then if $\langle v, Tv\rangle = 0$ for all $v \in V$, then $T = 0$.*

Proof. The idea is to show something even more general, namely that $\langle u, Tv\rangle = 0$ for all $u, v \in V$. If we can do this, then setting $u = Tv$ means that $\langle Tv, Tv\rangle = 0$ for all $v \in V$, which in turn implies that $Tv = 0$ for all $v \in V$, implying further that $T = 0$.

Zwiebach goes through a few stages in developing the proof, but the end result is that we can write

$$\begin{aligned}\langle u, Tv\rangle = \frac{1}{4}\left[\langle u+v, T(u+v)\rangle - \langle u-v, T(u-v)\rangle\right] + \\ \frac{1}{4i}\left[\langle u+iv, T(u+iv)\rangle - \langle u-iv, T(u-iv)\rangle\right] \end{aligned} \tag{1}$$

You can verify this by multiplying out the RHS. Note that all the terms on the RHS are of the form $\langle x, Tx\rangle$ for some x. Thus if we require $\langle x, Tx\rangle = 0$ for all $x \in V$, then all four terms are separately 0, meaning that $\langle u, Tv\rangle = 0$ as desired, completing the proof. $\square$

Although we've used the imaginary number i in this proof, we might wonder if it really does restrict the result to complex vector spaces. That is, is there some other decomposition of $\langle u, Tv\rangle$ that *doesn't* required complex numbers that would still work?

In fact, we don't need to worry about this, since there is a simple counter-example to the theorem if we consider a real vector space. In 2-d or 3-d space, an operator T that rotates a vector through $\frac{\pi}{2}$ always produces a vector orthogonal to the original, resulting in $\langle v, Tv \rangle = 0$ for all v. In this case, $T \neq 0$ so the theorem is definitely *not* true for real vector spaces.

Now we can turn to a few theorems about hermitian operators. First, since every operator on a finite-dimensional complex vector space has at least one eigenvalue, we know that every hermitian operator has at least one eigenvalue. This leads to the first theorem on hermitian operators.

Theorem 2. *All eigenvalues of hermitian operators are real.*

Proof. Since at least one eigenvalue λ exists, let v be the corresponding non-zero eigenvector, so that $Tv = \lambda v$. We have

$$\langle v, Tv \rangle = \langle v, \lambda v \rangle = \lambda \langle v, v \rangle \tag{2}$$

Since $T = T^\dagger$ we also have

$$\langle v, Tv \rangle = \left\langle T^\dagger v, v \right\rangle = \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda^* \langle v, v \rangle \tag{3}$$

Equating the last two equations, and remembering that $\langle v, v \rangle \neq 0$, we have $\lambda = \lambda^*$, so λ is real. □

This theorem has an important application to quantum mechanics, where it turns out that any operator that represents a physically measurable quantity is hermitian, meaning that it has real eigenvalues. Physical quantities must, of course, have real values, not complex ones.

Next, a theorem on the eigenvectors of distinct eigenvalues.

Theorem 3. *Eigenvectors associated with different eigenvalues of a hermitian operator are orthogonal.*

Proof. Suppose $\lambda_1 \neq \lambda_2$ are two eigenvalues of T, and v_1 and v_2 are the corresponding eigenvectors. Then $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$. Taking an inner product, we have

$$\begin{aligned} \langle v_2, Tv_1 \rangle &= \lambda_1 \langle v_2, v_1 \rangle & (4) \\ \langle v_2, Tv_1 \rangle &= \langle Tv_2, v_1 \rangle & (5) \\ &= \lambda_2 \langle v_2, v_1 \rangle & (6) \end{aligned}$$

where in the last line we used the fact that λ_2 is real when taking it outside the inner product. Equating the first and last lines and using $\lambda_1 \neq \lambda_2$, we see that $\langle v_2, v_1 \rangle = 0$ as required. □

References

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