HYPERGEOMETRIC SERIES

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Hypergeometric series are used in the solutions of certain differential equations. We consider two types of series here.

Gaussian hypergeometric series. We define this series as

$$ {}_2F_1(b,c,d;z) \equiv \sum_{j=0}^{\infty} \frac{(b)_j (c)_j}{(d)_j} \frac{z^j}{j!} \tag{1}$$

where the symbols in parentheses are defined as

$$(a)_j \equiv a(a+1)\dots(a+j-1) \tag{2}$$

for any complex number a, with $(a)_0 = 1$. Note that because a can be any complex number, $(a)_j$ is not always just a ratio of factorials.

We first find the radius of convergence of 1 using the test for convergence of power series, namely if the ratio of coefficients has a limit

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L \tag{3}$$

then the radius of convergence is given by

$$R = \frac{1}{L} \tag{4}$$

For 1 we have

$$\begin{aligned}
\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| &= \lim_{j\to\infty} \left| \frac{(b)_{j+1}(c)_{j+1}}{(d)_{j+1}(j+1)!} \times \frac{(d)_j\, j!}{(b)_j (c)_j} \right| &(5)\\
&= \lim_{j\to\infty} \left| \frac{(b+j)(c+j)}{(d+j)(j+1)} \right| &(6)\\
\lim_{j\to\infty} &\left| \frac{j^2+(b+c)j+bc}{j^2+(d+1)j+j} \right| &(7)\\
&= 1 &(8)
\end{aligned}$$

where the last line follows because the leading term in numerator and denominator is j^2. Thus the radius of convergence is

$$R = 1 \tag{9}$$

and the series converges for $|z| < 1$.

The differential equation satisfied by 1 is

$$z(1-z)f'' + [d - (b+c+1)z]f' - bcf = 0 \tag{10}$$

In order to simplify the notation I'll omit the parentheses from terms such $(a)_j$ and so, and just call these quantities a_j. A quantity a without a subscript refers to the complex number a in 2.

Substituting the series into 10 gives us

$$z(1-z)\sum_{j=2}^{\infty}\frac{b_jc_j}{d_j}\frac{z^{j-2}}{(j-2)!} + [d-(b+c+1)z]\sum_{j=1}^{\infty}\frac{b_jc_j}{d_j}\frac{z^{j-1}}{(j-1)!} - bc\sum_{j=0}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{j!} \tag{11}$$

Note the lower limits on the sums have been adjusted so the lowest exponent is zero in each sum.

Multiplying through and collecting terms, we have

$$-\sum_{j=2}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{(j-2)!} - (b+c+1)\sum_{j=1}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{(j-1)!} - bc\sum_{j=0}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{j!} + \sum_{j=2}^{\infty}\frac{b_jc_j}{d_j}\frac{z^{j-1}}{(j-2)!} + d\sum_{j=1}^{\infty}\frac{b_jc_j}{d_j}\frac{z^{j-1}}{(j-1)!} \tag{12}$$

We can adjust the summation index on the last two terms so that both terms have a z^j factor. This gives

$$-\sum_{j=2}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{(j-2)!} - (b+c+1)\sum_{j=1}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{(j-1)!} - bc\sum_{j=0}^{\infty}\frac{b_jc_j}{d_j}\frac{z^j}{j!} + \sum_{j=1}^{\infty}\frac{b_{j+1}c_{j+1}}{d_{j+1}}\frac{z^j}{(j-1)!} + d\sum_{j=0}^{\infty}\frac{b_{j+1}c_{j+1}}{d_{j+1}}\frac{z^j}{j!} \tag{13}$$

We now need to show that the coefficient of each power of z is zero individually. First, we consider $j \geq 2$ since this gets a contribution from all sums. We have for the coefficient of z^j:

$$\frac{b_j c_j}{d_j}\left[-\frac{1}{(j-2)!}-\frac{b+c+1}{(j-1)!}-\frac{bc}{j!}+\frac{(b+j)(c+j)}{(d+j)}\left(\frac{1}{(j-1)!}+\frac{d}{j!}\right)\right] \tag{14}$$

Entering this expression into Maple confirms that it is zero for all $j \geq 2$. If you want to grind through the algebra, note that

$$\frac{(b+j)(c+j)}{(d+j)}\left(\frac{1}{(j-1)!}+\frac{d}{j!}\right) = \frac{(b+j)(c+j)}{(d+j)}\left(\frac{d+j}{j!}\right) \tag{15}$$

$$= \frac{(b+j)(c+j)}{j!} \tag{16}$$

and

$$-\frac{1}{(j-2)!}-\frac{b+c+1}{(j-1)!}-\frac{bc}{j!} = -\frac{j(j-1)+(b+c+1)j-bc}{j!} \tag{17}$$

$$= -\frac{(b+j)(c+j)}{j!} \tag{18}$$

Thus the result follows.

To complete the demonstration, we need to show that the terms for $j=0$ and $j=1$ also give zero. For $j=0$ we have, from the third and last terms in 13

$$-bc\frac{b_0 c_0}{d_0}\frac{z^0}{0!} = -bc \tag{19}$$

$$d\frac{b_1 c_1}{d_1}\frac{z^0}{0!} = d\frac{bc}{d} = bc \tag{20}$$

Thus the terms for $j=0$ give zero. For $j=1$ we have from the last 4 terms in 13 the coefficients of z^1:

$$-(b+c+1)\frac{b_1 c_1}{d_1 0!} = -\frac{(b+c+1)bc}{d} \tag{21}$$

$$-bc\frac{b_1 c_1}{d_1 1!} = -\frac{b^2 c^2}{d} \tag{22}$$

$$\frac{b_2 c_2}{d_2 0!} = \frac{b(b+1)c(c+1)}{d(d+1)} \tag{23}$$

$$d\frac{b_2 c_2}{d_2 1!} = \frac{b(b+1)c(c+1)}{d+1} \tag{24}$$

Again, you can grind through the algebra if you like, but I used Maple to verify that the sum of these 4 terms is indeed zero. Thus 13 does indeed solve 10.

Confluent hypergeometric series. The second series we'll consider is

$$ {}_1F_1(c,d;z) \equiv \sum_{j=0}^{\infty} \frac{(c)_j}{(d)_j} \frac{z^j}{j!} \tag{25} $$

First, we'll look at the radius of convergence. Using the same technique as above, we have

$$ \lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \left| \frac{(c)_{j+1}}{(d)_{j+1}(j+1)!} \times \frac{(d)_j j!}{(c)_j} \right| \tag{26} $$

$$ = \lim_{j\to\infty} \left| \frac{c+j}{(d+j)(j+1)} \right| \tag{27} $$

$$ = 0 \tag{28} $$

Thus the radius of convergence is $R = \infty$ and the series converges for all z.

The differential equation satisfied by 25 is

$$ zf'' + (d-z) f' - cf = 0 \tag{29} $$

Substituting the series 25 into this gives

$$ z \sum_{j=2}^{\infty} \frac{c_j}{d_j} \frac{z^{j-2}}{(j-2)!} + (d-z) \sum_{j=1}^{\infty} \frac{c_j}{d_j} \frac{z^{j-1}}{(j-1)!} - c \sum_{j=0}^{\infty} \frac{c_j}{d_j} \frac{z^j}{j!} \tag{30} $$

Multiplying through and collecting powers of z gives

$$ \sum_{j=2}^{\infty} \frac{c_j}{d_j} \frac{z^{j-1}}{(j-2)!} + d \sum_{j=1}^{\infty} \frac{c_j}{d_j} \frac{z^{j-1}}{(j-1)!} - \sum_{j=1}^{\infty} \frac{c_j}{d_j} \frac{z^j}{(j-1)!} - c \sum_{j=0}^{\infty} \frac{c_j}{d_j} \frac{z^j}{j!} \tag{31} $$

Adjusting the summation index on the first two terms gives

$$ \sum_{j=1}^{\infty} \frac{c_{j+1}}{d_{j+1}} \frac{z^j}{(j-1)!} + d \sum_{j=0}^{\infty} \frac{c_{j+1}}{d_{j+1}} \frac{z^j}{j!} - \sum_{j=1}^{\infty} \frac{c_j}{d_j} \frac{z^j}{(j-1)!} - c \sum_{j=0}^{\infty} \frac{c_j}{d_j} \frac{z^j}{j!} \tag{32} $$

All terms contribute for $j \geq 1$ so in this case we have

$$\frac{c_j}{d_j}\left[\frac{c+j}{d+j}\frac{1}{(j-1)!}+\frac{c+j}{d+j}\frac{d}{j!}-\frac{1}{(j-1)!}-\frac{c}{j!}\right]= \tag{33}$$

$$\frac{c_j}{d_j}\left[\frac{c+j}{d+j}\frac{d+j}{j!}-\frac{c+j}{j!}\right]=0 \tag{34}$$

Finally, we need to verify that the coefficient for $j=0$ in 32 is zero as well. We have from the second and last terms

$$d\frac{c_1}{d_1}\frac{z^0}{0!}-c\frac{c_0}{d_0}\frac{z^0}{0!}=d\frac{c}{d}-c=0 \tag{35}$$