IMPROPER INTEGRALS AND JORDAN'S LEMMA

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The earlier examples of improper integrals involving trig functions and rational polynomials relied on the degree of the polynomial in the denominator being at least 2 greater than the degree of the polynomial in the numerator. This ensured that the integral over the semicircular arc in either the upper or lower half plane went to zero as the arc went to infinity.

However, there is a useful theorem known as *Jordan's lemma* that allows us to calculate integrals where the degree of the denominator is only 1 greater than that of the numerator. It states:

Lemma 1. *Jordan's lemma. If $m>0$ and P/Q is the quotient of two polynomials such that the degree of $Q \geq$ degree of P, then*

$$\lim_{\rho\to\infty}\int_{C_\rho^+} e^{imz}\frac{P(z)}{Q(z)}dz = 0 \tag{1}$$

where C_ρ^+ is the semicircular arc in the upper half plane, of radius ρ.

A corollary is that

$$\lim_{\rho\to\infty}\int_{C_\rho^-} e^{-imz}\frac{P(z)}{Q(z)}dz = 0 \tag{2}$$

where C_ρ^- is the semicircular arc in the lower half plane.

Example 1. Find

$$I = \int_{-\infty}^{\infty}\frac{e^{3ix}}{x-2i}dx \tag{3}$$

There is a single pole in the upper half plane at $x=2i$. The residue is

$$\text{Res}(2i) = e^{-6} \tag{4}$$

so we have

$$I = 2\pi i e^{-6} \tag{5}$$

Example 2. Find

$$I=\int_0^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}dx \tag{6}$$

We note that the numerator is the product of two odd functions and the denominator is even, so the integrand as a whole is even. Therefore

$$I=\frac{1}{2}\int_{-\infty}^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}dx \tag{7}$$

To use Jordan's lemma, we use

$$\sin(2x)=\frac{e^{2ix}-e^{-2ix}}{2i} \tag{8}$$

and write this as

$$I=I_1+I_2 \tag{9}$$

$$=\frac{1}{4i}\int_{-\infty}^\infty \frac{x^3e^{2ix}}{(x^2+1)^2}dx-\frac{1}{4i}\int_{-\infty}^\infty \frac{x^3e^{-2ix}}{(x^2+1)^2}dx \tag{10}$$

There are poles at $x=\pm i$. For I_1 we need only the pole in the upper half plane, at $x=i$. We have

$$I_1=\frac{1}{4i}\int_{-\infty}^\infty \frac{x^3e^{2ix}}{(x+i)^2(x-i)^2}dx \tag{11}$$

$$\text{Res}\left(\frac{x^3e^{2ix}}{(x^2+1)^2},i\right)=\lim_{x\to i}\frac{d}{dx}\left[(x-i)^2\frac{x^3e^{2ix}}{(x^2+1)^2}\right] \tag{12}$$

Using Maple to do the calculations, we find

$$\text{Res}\left(\frac{x^3e^{2ix}}{(x^2+1)^2},i\right)=\lim_{x\to i}\frac{3(x-i)e^{2ix}x^2\left(x^3+\frac{2ix^4}{3}+\frac{7x}{3}+ix^2-i\right)}{(x^2+1)^3} \tag{13}$$

$$=0 \tag{14}$$

Similarly, we find

$$\text{Res}\left(\frac{x^3e^{-2ix}}{(x^2+1)^2},-i\right)=\lim_{x\to -i}\frac{d}{dx}\left[(x+i)^2\frac{x^3e^{-2ix}}{(x^2+1)^2}\right] \tag{15}$$

$$=0 \tag{16}$$

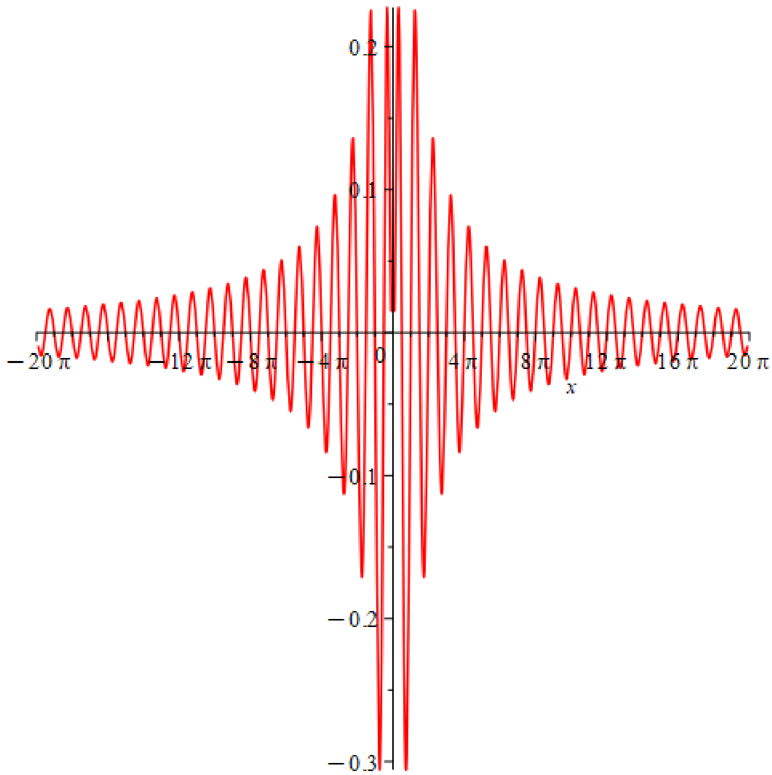

FIGURE 1. Plot of $\frac{x^3 \sin(2x)}{\left(x^2+1\right)^2}$ in the range $[-20\pi, 20\pi]$.

Thus both residues are zero, so

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^3 \sin(2x)}{\left(x^2+1\right)^2} dx = 0 \tag{17}$$

If this seems unlikely, I checked it by doing the integral numerically in Maple, with the result $I = -5 \times 10^{-14}$, so it looks right. A graph of the integrand is shown in Fig. 1 for the range $x \in [-20\pi, 20\pi]$, and it looks as though the function spends about as much time below the x axis as above it, so the result is reasonable.

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