IMPROPER INTEGRALS OF RATIONAL FUNCTIONS

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Post date: 1 April 2025.

An *improper integral* is an integral where one or both of its limits are infinite. Technically, such integrals are defined by a limit, as in

$$\int_{-\infty}^{\infty}\frac{1}{x^2}dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{1}{x^2+1}dx \tag{1}$$

$$= \lim_{\rho\to\infty}\arctan x\Big|_{-\rho}^{\rho} \tag{2}$$

$$= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) \tag{3}$$

$$= \pi \tag{4}$$

Some functions have such improper integrals, although the integral over half the range may not be defined. For example

$$\int_{-\infty}^{\infty}x\,dx = \lim_{\rho\to\infty}\left.\frac{x^2}{2}\right|_{-\rho}^{\rho} = 0 \tag{5}$$

However, the integral over the range $[0,\infty]$ gives an infinite value from the upper limit. If the limit over the interval $(-\infty,\infty)$ (if it exists) is called the *Cauchy principal value*, and is often denoted by writing 'p.v.' before the integral, as in

$$\text{p.v.}\int_{-\infty}^{\infty}x\,dx = 0 \tag{6}$$

In what follows, we'll assume that the principal value is meant whenever the integral is over $(-\infty,\infty)$.

We can use contour integration to find the values of many infinite integrals. In this post, we'll be concerned with rational functions (the ratio of two polynomials) where the degree of the denominator is at least 2 greater than the degree of the numerator. That is, we consider

$$f(z) = \frac{P(z)}{Q(z)} \tag{7}$$

Further, we'll assume that the all the denominator's zeroes lie off the real axis. [We'll relax this restriction in a future post.]

The contour C we use extends along the real axis, initially over the interval $[-\rho,\rho]$, where we take the limit as $\rho\to\infty$ at the end of the calculation. The contour is closed by drawing a semicircular arc of radius ρ, extending from $x=\rho$ on the right, up through $y=\rho i$ on the imaginary axis, and ending at $x=-\rho$ on the left. Since the denominator has no zeroes on the real axis, this contour does not intersect any poles of the function (or at least it won't once we let ρ go to infinity). The value of the contour integral is therefore

$$\oint_C \frac{P(z)}{Q(z)}dz = 2\pi i\sum_k \text{Res}(z_k) \tag{8}$$

where z_k are all the poles in the upper half plane. Poles in the lower half plane lie outside the contour, so are not included. Saff and Snider prove that the integral of 7 over the semicircular arc is zero for rational functions satisfying the above conditions. Basically this is because, as we let $\rho\to\infty$, the function $f(z)$ goes to zero at order ρ^{m-n} (where m is the degree of P and n is the degree of Q, so that $n\ge m+2$), and the length of the semicircular arc goes as $\pi\rho$ so the overall integral over the arc tends to zero at order ρ^{m-n+1}, and $m-n+1\le -1$.

Thus the calculation reduces to finding the zeroes z_k of $Q(z)$, and then finding the residue at each of these points.

Example 1. Given

$$\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}dx \tag{9}$$

Using the quadratic formula, the zeroes of $Q(x)$ are at

$$z_k = -1+i, -1-i \tag{10}$$

Only $-1+i$ lies within the contour. The residue is

$$\text{Res}(-1+i) = -\frac{i}{2} \tag{11}$$

Thus we have

$$\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}dx = 2\pi i\left(-\frac{i}{2}\right) = \pi \tag{12}$$

Example 2. Given

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+9\right)^2} dx \tag{13}$$

The poles are of order 2 and are at

$$x = \pm 3i \tag{14}$$

Only $+3i$ lies within the contour. We have

$$\text{Res}\,(3i) = -\frac{i}{12} \tag{15}$$

so

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(x^2+9\right)^2} dx = 2\pi i \left(-\frac{i}{12}\right) \tag{16}$$

$$= \frac{\pi}{6} \tag{17}$$

Example 3. Given

$$\int_{0}^{\infty} \frac{x^2+1}{x^4+1} dx \tag{18}$$

This time, the lower limit is 0 rather than $-\infty$, but we observe that the integrand is an even function, so we have

$$\int_{0}^{\infty} \frac{x^2+1}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} dx \tag{19}$$

The poles are at the fourth roots of 1. Only two of these have positive imaginary parts, and they are

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2} \tag{20}$$

The residue calculations are a bit messy, but since the poles are order 1, probably the easiest is to use the formula

$$\text{Res}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}\right) = \left.\frac{P(x)}{Q'(x)}\right|_{x=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}} \tag{21}$$

$$= \left.\frac{x^2+1}{4x^3}\right|_{x=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}} \tag{22}$$

$$= -\frac{\sqrt{2}}{4}i \tag{23}$$

The other residue is the same:

$$\text{Res}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right) = -\frac{\sqrt{2}}{4}i \tag{24}$$

so we have

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}2\pi i\left(-\frac{\sqrt{2}}{4}i-\frac{\sqrt{2}}{4}i\right) \tag{25}$$

$$= \frac{\sqrt{2}\pi}{2} \tag{26}$$

Example 4. Given

$$\int_{-\infty}^\infty \frac{x}{\left(x^2+4x+13\right)^2}dx \tag{27}$$

The poles are of order 2, and are at

$$x = -2+3i, -2-3i \tag{28}$$

Only $-2+3i$ lies within the contour. The residue is quite messy so I used Maple to do the derivative and algebra.

$$\text{Res}\left(-2+3i\right) = \lim_{x\to -2+3i}\left[\frac{d}{dx}\left(\left(x-\left(-2+3i\right)\right)^2\frac{x}{\left(x^2+4x+13\right)^2}\right)\right] \tag{29}$$

$$= \frac{i}{54} \tag{30}$$

Thus we have

$$\int_{-\infty}^\infty \frac{x}{\left(x^2+4x+13\right)^2}dx = 2\pi i\left(\frac{i}{54}\right) = -\frac{\pi}{27} \tag{31}$$

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