IMPROPER INTEGRALS USING LOGS 2

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 30 April 2025.

Here's another example of using logarithms to calculate improper integrals.

Find

$$I=\int_0^\infty \frac{1}{x^3+1}dx \tag{1}$$

As before, the integrand is not an even function, so we introduce logarithms to allow integration around a branch cut. That is, we consider

$$I_L=\int_0^\infty \frac{\mathcal{L}_0(x)}{x^3+1}dx \tag{2}$$

where

$$\mathcal{L}_0(x)=\text{Log}\,|x|+i\arg x \tag{3}$$

$$=\text{Log}\,|x|+i\theta \tag{4}$$

where $0\le\theta<2\pi$. That is, we've introduced the logarithm with a branch cut along the non-negative real axis. We now consider the contour integral of 2 around the contour shown in Fig. 1.

The denominator factors into the three cube roots of -1, so we have

$$I_L=\int_0^\infty \frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)} \tag{5}$$

As with our previous example, the contour integral of 2 is

$$\left[\int_{C_\rho}+\int_{\Gamma_\varepsilon}+\int_{\gamma_1}+\int_{\gamma_2}\right]\frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)}dx= \tag{6}$$

$$\left[\int_{C_\rho}+\int_{\Gamma_\varepsilon}\right]\frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)}dx+ \tag{7}$$

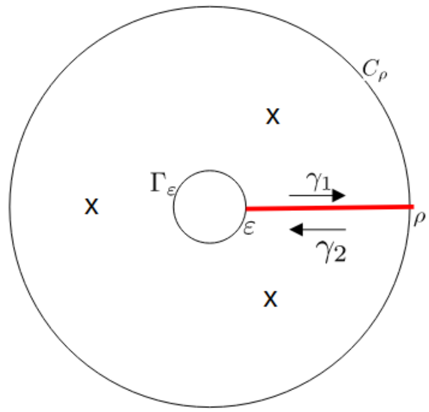

FIGURE 1. Contour, with poles indicated by x.

$$-2\pi i \int_{\varepsilon}^{\rho} \frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)} dx = 2\pi i \sum(\text{residues}) \tag{8}$$

We note here that, since all the cube roots of -1 have magnitude 1, in 4 $\text{Log}\,|x| = 0$ for all roots.

If we can show that the integrals around the two circles go to zero in the limit, we'll have a formula for our desired integral 1.

Consider the integral around Γ_ε. We have, multiplying the integrand by the circumference of the circle ($2\pi\varepsilon$) and taking the limit as $\varepsilon \to 0$:

$$\left| \int_{\Gamma_\varepsilon} \frac{\mathcal{L}_0(x)}{x^3+1} dx \right| \leq \frac{2\pi\varepsilon \times |\text{Log}\varepsilon + 2\pi|}{1-\varepsilon^3} \to 0 \tag{9}$$

For C_ρ, we have in the limit $\rho \to \infty$:

$$\left| \int_{C_\rho} \frac{\mathcal{L}_0(x)}{x^3+1} dx \right| \leq \frac{2\pi\rho \times |\text{Log}\rho + 2\pi|}{\rho^3 - 1} \to 0 \tag{10}$$

Therefore, from 8, in the limit $\varepsilon \to 0$ and $\rho \to \infty$:

$$\int_{0}^{\infty} \frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)} dx = -\sum(\text{residues}) \tag{11}$$

Now to calculate the residues. The argument θ in 4 is in the range $0 \leq \theta \leq 2\pi$, so we have $\theta = \pi$ for $x = -1$, $\theta = \pi/3$ for $x = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\theta = 5\pi/3$ for $x = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

$$\operatorname{Res}(-1) = (x+1)\left.\frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)}\right|_{x=-1} \tag{12}$$

$$= \frac{i\pi}{\left(-\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)} \tag{13}$$

$$\operatorname{Res}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) = \left(x-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\left.\frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)}\right|_{x=\frac{1}{2}+\frac{\sqrt{3}i}{2}} \tag{14}$$

$$= \frac{i\pi/3}{\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)\left(i\sqrt{3}\right)} \tag{15}$$

$$\operatorname{Res}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right) = \left(x-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\left.\frac{\mathcal{L}_0(x)}{(x+1)\left(x-\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\right)\left(x-\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\right)}\right|_{x=\frac{1}{2}-\frac{\sqrt{3}i}{2}} \tag{16}$$

$$= \frac{i5\pi/3}{\left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)\left(-i\sqrt{3}\right)} \tag{17}$$

Adding 13, 15 and 17 (using Maple to do the arithmetic), we get

$$\sum(\text{residues}) = -\frac{2\pi\sqrt{3}}{9} \tag{18}$$

Therefore

$$\int_0^\infty \frac{1}{x^3+1}dx = -\sum(\text{residues}) = \frac{2\pi\sqrt{3}}{9} \approx 1.2092 \tag{19}$$

A plot of the integrand in 1 is shown in Fig. 2.

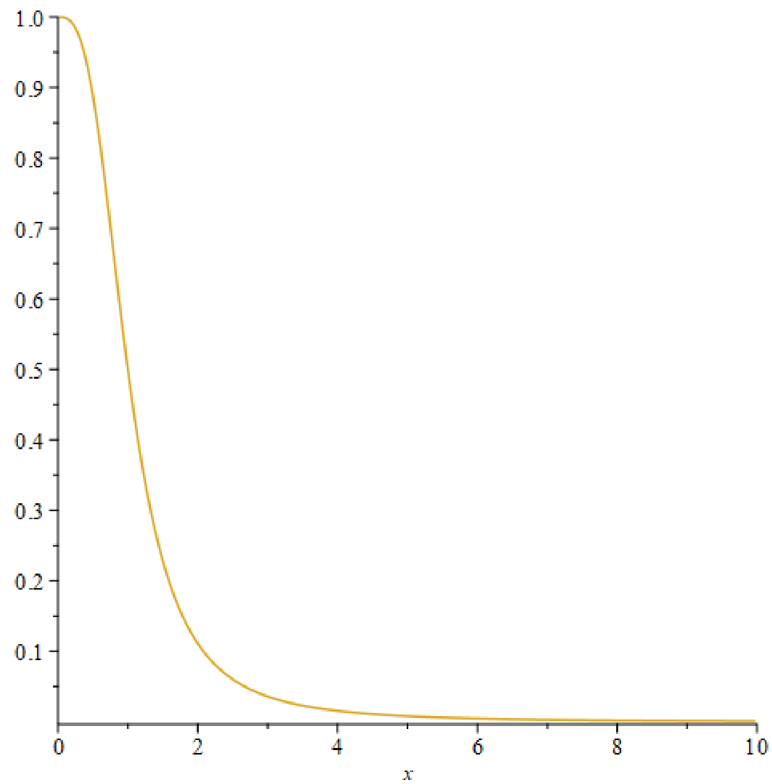

FIGURE 2. Plot of $\frac{1}{x^3+1}$.