IMPROPER INTEGRALS USING LOGS

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Here's another technique for using residue theory to calculate integrals of real functions.

Find

$$I = \int_0^\infty \frac{x}{(x+1)(x^2+2x+2)}dx \tag{1}$$
$$= \int_0^\infty \frac{x}{(x+1)(x-(-1+i))(x-(-1-i))}dx \tag{2}$$

where we've factored the denominator to get the last line.

At first glance, we might be tempted to try defining a semicircle with edge along the real axis and let the radius tend to infinity, as we've used before. The problem here is that we are integrating from 0 to ∞ rather than from $-\infty$ to ∞, and the integrand isn't an even function, so we can't just take half the integral from $-\infty$ to ∞. Instead, we use a trick involving integrating over a contour that includes a branch cut.

The integrand as it stands is analytic everywhere except at the poles in the denominator, so it doesn't have any branch cuts. However, we can introduce the Log function, which does have branch cuts. That is, we consider

$$I_L = \int_0^\infty \frac{x\mathcal{L}_0(x)}{(x+1)(x-(-1+i))(x-(-1-i))}dx \tag{3}$$

where

$$\mathcal{L}_0(x) = \text{Log}|x| + i\arg x \tag{4}$$
$$= \text{Log}|x| + i\theta \tag{5}$$

where $0 \le \theta < 2\pi$. That is, we've introduced the logarithm with a branch cut along the non-negative real axis. We now consider the contour integral of 3 around the contour shown in Fig. 1.

We might think that introducing the logarithm just makes the problem worse, but consider the integrals along the straight line segments γ_1 and γ_2. Along γ_1, $\theta = 0$ so $\mathcal{L}_0(x) = \text{Log}|x|$. Along γ_2, $\theta = 2\pi$, so $\mathcal{L}_0(x) =$

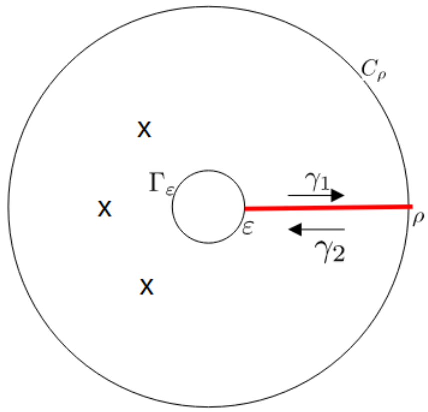

FIGURE 1. Contour for integrating 3, with poles at points marked x.

$\operatorname{Log}|x| + 2\pi i$. Since the direction of integration along γ_1 is opposite to that along γ_2, the integrals involving $\operatorname{Log}|x|$ cancel. Thus we have

$$\left[\int_{C_\rho} + \int_{\Gamma_\varepsilon} + \int_{\gamma_1} + \int_{\gamma_2}\right] \frac{x\mathcal{L}_0(x)}{(x+1)(x-(-1+i))(x-(-1-i))} dx = \tag{6}$$

$$\left[\int_{C_\rho} + \int_{\Gamma_\varepsilon}\right] \frac{x\mathcal{L}_0(x)}{(x+1)(x-(-1+i))(x-(-1-i))} dx + \tag{7}$$

$$\int_\varepsilon^\rho \frac{x\operatorname{Log}|x|}{(x+1)(x-(-1+i))(x-(-1-i))} dx + \tag{8}$$

$$\int_\rho^\varepsilon \frac{x(\operatorname{Log}|x| + 2\pi i)}{(x+1)(x-(-1+i))(x-(-1-i))} dx = 2\pi i \sum(\text{residues}) \tag{9}$$

Cancelling the integrals in the last two lines, we have

$$\left[\int_{C_\rho} + \int_{\Gamma_\varepsilon} + \int_{\gamma_1} + \int_{\gamma_2}\right] \frac{x\mathcal{L}_0(x)}{(x+1)(x-(-1+i))(x-(-1-i))} dx = \tag{10}$$

$$\left[\int_{C_\rho} + \int_{\Gamma_\varepsilon}\right] \frac{x\mathcal{L}_0(x)}{(x+1)(x-(-1+i))(x-(-1-i))} dx + \tag{11}$$

$$-2\pi i \int_\varepsilon^\rho \frac{x}{(x+1)(x-(-1+i))(x-(-1-i))} dx = 2\pi i \sum(\text{residues}) \tag{12}$$

If we can show that the integrals around the two circles go to zero in the limit, we'll have a formula for our desired integral 2.

Consider the integral around Γ_ε. We have, multiplying the integrand by the circumference of the circle ($2\pi\varepsilon$):

$$\left|\int_{\Gamma_\varepsilon} \frac{x\mathcal{L}_0(x)}{(x+1)(x^2+2x+2)}dx\right| \le \frac{2\pi\varepsilon\times\varepsilon\,|\mathrm{Log}\,|\varepsilon|+2\pi i|}{(1-\varepsilon)(2-2\varepsilon-\varepsilon^2)} \tag{13}$$

As $\varepsilon\to 0$, the numerator goes as $\varepsilon\mathrm{Log}\varepsilon$. Although $\mathrm{Log}\varepsilon\to-\infty$ as $\varepsilon\to 0$, it does so more slowly than $\varepsilon^2\to 0$, so the product $\varepsilon^2\mathrm{Log}\varepsilon\to 0$.

For C_ρ, we have

$$\left|\int_{C_\rho} \frac{x\mathcal{L}_0(x)}{(x+1)(x^2+2x+2)}dx\right| \le \frac{2\pi\rho\times\rho\,|\mathrm{Log}\rho+2\pi i|}{(\rho-1)(\rho^2-2\rho-2)} \tag{14}$$

$$= \frac{2\pi\rho^2\,|\mathrm{Log}\rho+2\pi i|}{\rho^3(1-1/\rho)(1-2/\rho-2/\rho^2)} \tag{15}$$

The ratio goes as $\left(\rho^2\mathrm{Log}\rho\right)/\rho^3$ for large ρ, which again goes to zero, since $\mathrm{Log}\rho$ increases more slowly than ρ.

Therefore in the limit as $\varepsilon\to 0$ and $\rho\to\infty$, we have from 12

$$\int_0^\infty \frac{x}{(x+1)(x-(-1+i))(x-(-1-i))}dx = -\sum(\text{residues}) \tag{16}$$

There are 3 simple poles, at $x=-1,-1\pm i$. The values of the argument θ are within the range $0\le\theta\le 2\pi$, and are $\theta=\pi$ for $x=-1$, $\theta=3\pi/4$ for $x=-1+i$ and $\theta=5\pi/4$ for $x=-1-i$. We have

$$\mathrm{Res}(-1) = (x+1)\left.\frac{x[\mathrm{Log}\,|x|+i\theta]}{(x+1)(x-(-1+i))(x-(-1-i))}\right|_{x=-1} \tag{17}$$

$$= \frac{-1\times\pi i}{-i\times i} \tag{18}$$

$$= -\pi i \tag{19}$$

$$\mathrm{Res}(-1+i) = (x-(-1+i))\left.\frac{x[\mathrm{Log}\,|x|+i\theta]}{(x+1)(x-(-1+i))(x-(-1-i))}\right|_{x=-1+i} \tag{20}$$

$$= \frac{(-1+i)\left(\mathrm{Log}\left(\sqrt{2}\right)+3\pi i/4\right)}{i\times 2i} \tag{21}$$

$$= -\frac{1}{2}(-1+i)\left(\mathrm{Log}\left(\sqrt{2}\right)+3\pi i/4\right) \tag{22}$$

$$\mathrm{Res}(-1-i) = (x-(-1-i))\left.\frac{x[\mathrm{Log}\,|x|+i\theta]}{(x+1)(x-(-1+i))(x-(-1-i))}\right|_{x=-1-i} \tag{23}$$

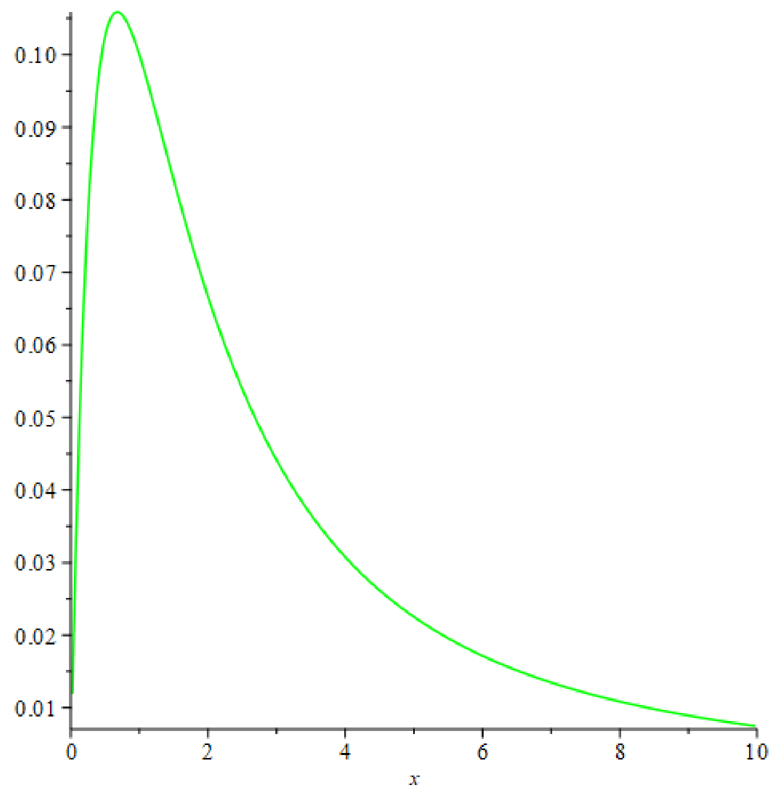

FIGURE 2. Plot of $\frac{x}{(x+1)\left(x^2+2x+2\right)}$.

$$= \frac{(-1-i)\left(\text{Log}\left(\sqrt{2}\right)+5\pi i/4\right)}{-i\times(-2i)} \tag{24}$$

$$= -\frac{1}{2}(-1-i)\left(\text{Log}\left(\sqrt{2}\right)+5\pi i/4\right) \tag{25}$$

Adding up 19, 22 and 25 (using Maple to do the arithmetic) and inserting into 16 we have

$$\int_0^\infty \frac{x}{(x+1)\left(x^2+2x+2\right)}dx = -\left[\text{Log}\left(\sqrt{2}\right)-\frac{\pi}{4}\right] \tag{26}$$

$$= -\text{Log}\left(\sqrt{2}\right)+\frac{\pi}{4} \approx 0.4388 \tag{27}$$

A plot of the integrand is shown in Fig. 2.

PINGBACKS

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