IMPROPER INTEGRALS WITH INDENTED CONTOURS EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 19 April 2025.

Here are a few more examples of improper integrals with indented contours.

Example 1. Find

$$I=\int_{0}^{\infty}\frac{\cos x-1}{x^{2}}dx \tag{1}$$

The integrand is even, so we have

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x-1}{x^{2}}dx \tag{2}$$

Expanding $\cos x$ in a series, we get

$$\frac{\cos x-1}{x^{2}}=\frac{1}{x^{2}}\left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\ldots-1\right) \tag{3}$$

$$=-\frac{1}{2}+\frac{x^{2}}{4!}-\ldots \tag{4}$$

Thus there is no pole at $x=0$. To fix this, we note that

$$I=\Re\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x^{2}}dx \tag{5}$$

which does have a pole at $x=0$. Expanding, we have

$$\frac{e^{ix}-1}{x^{2}}=\frac{1}{x^{2}}\left(1+ix+\frac{(ix)^{2}}{2!}+\ldots-1\right) \tag{6}$$

$$=\frac{i}{x}-\frac{1}{2}+\ldots \tag{7}$$

Thus

$$\text{Res}(0)=i \tag{8}$$

The integral around the indentation is thus

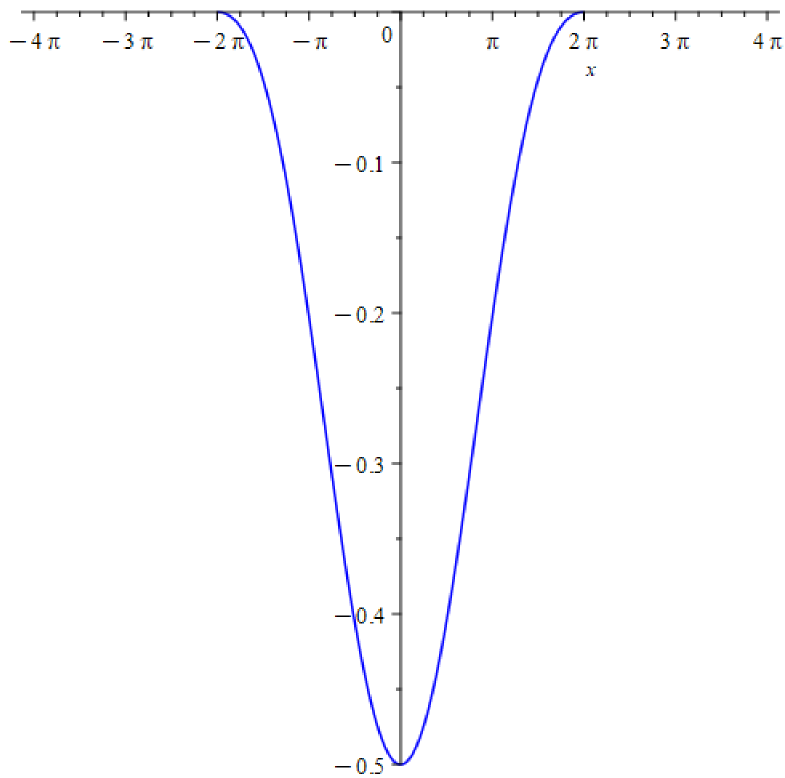

FIGURE 1. Plot of $\frac{\cos x-1}{x^2}$.

$$\lim_{r\to 0}\int_{S_r}\frac{e^{ix}-1}{x^2}dx=-i\pi i=\pi \tag{9}$$

and the integral around the contour in the upper half plane with an indentation around $z=0$ is zero, so

$$\lim_{r\to 0}\left[\int_{-\infty}^{r}+\int_{S_r}+\int_{r}^{\infty}\right]\frac{e^{ix}-1}{x^2}dx=0 \tag{10}$$

Therefore, since 9 is real

$$I=-\frac{1}{2}\Re\lim_{r\to 0}\int_{S_r}\frac{e^{ix}-1}{x^2}dx=-\frac{\pi}{2} \tag{11}$$

A plot is shown in Fig. 1.

Example 2. Find

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)} dx \tag{12}$$

We rewrite this as

$$I = \Im \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)} dx \tag{13}$$

There is a pole at $x = 1$ on the real axis, and a pole at $x = 2i$ in the upper half plane. The residues are

$$\begin{aligned} \text{Res}(1) &= \frac{e^i}{5} \\ \text{Res}(2i) &= \frac{1}{10}\left(-1+\frac{i}{2}\right)e^{-2} \end{aligned} \tag{14}$$

The integral around the entire contour is

$$\lim_{r\to 0}\left[\int_{-\infty}^{r} + \int_{S_r} + \int_{r}^{\infty} + \int_{C^+}\right] \frac{e^{ix}}{(x^2+4)(x-1)} dx = 2\pi i \text{Res}(2i) \tag{15}$$

$$= \frac{\pi}{5}\left(-i-\frac{1}{2}\right)e^{-2} \tag{16}$$

The integral around the indentation S_r is

$$\lim_{r\to 0}\int_{S_r} \frac{e^{ix}}{(x^2+4)(x-1)} dx = -i\pi \text{Res}(1) \tag{17}$$

$$= -i\frac{\pi}{5}e^i \tag{18}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)} dx = \frac{\pi}{5}\left(-i-\frac{1}{2}\right)e^{-2} + i\frac{\pi}{5}e^i \tag{19}$$

Using Maple to express this in the form $u + iv$, we have

$$\frac{\pi}{5}\left(-i-\frac{1}{2}\right)e^{-2} + i\frac{\pi}{5}e^i = -\frac{\pi}{5}\left(\frac{e^{-2}}{2}+\sin 1\right) + i\frac{\pi}{5}\left(-e^{-2}+\cos 1\right) \tag{20}$$

The integral I in 12 is the imaginary part, so

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)} dx = \frac{\pi}{5}\left(-e^{-2}+\cos 1\right) \approx 0.25445 \tag{21}$$

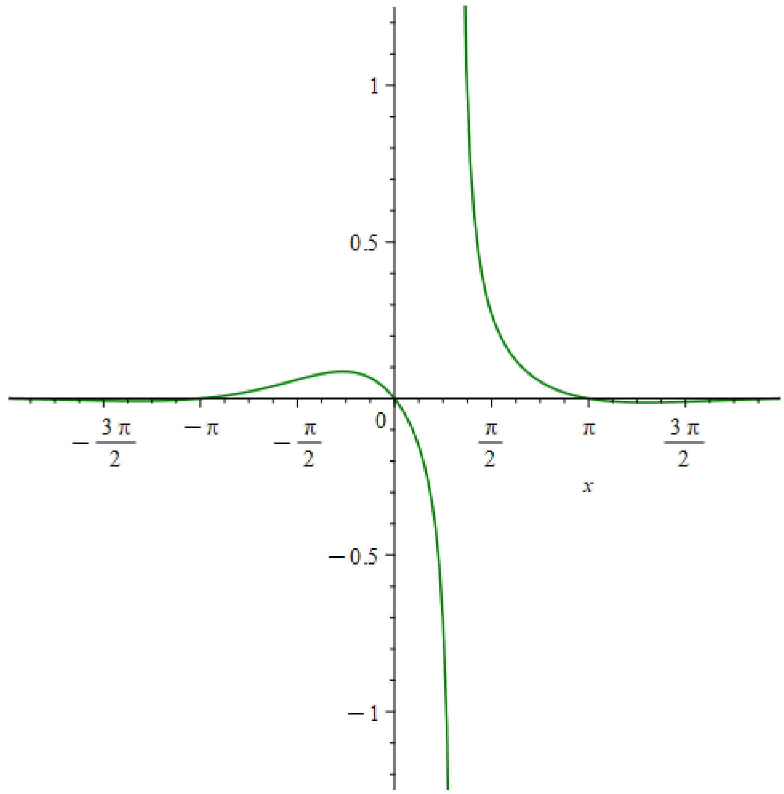

FIGURE 2. Plot of $\frac{\sin x}{(x^2+4)(x-1)}$.

As a bonus, by taking the real part, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x-1)} dx = -\frac{\pi}{5}\left(\frac{e^{-2}}{2} + \sin 1\right) \approx -0.57123 \tag{22}$$

A plot of the integrand in 12 is in Fig. 2, and of the integrand in 22 in Fig. 3.

Example 3. Find

$$I = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 3x + 2} dx \tag{23}$$

This is equivalent to

$$I = \Re \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 - 3x + 2} dx \tag{24}$$

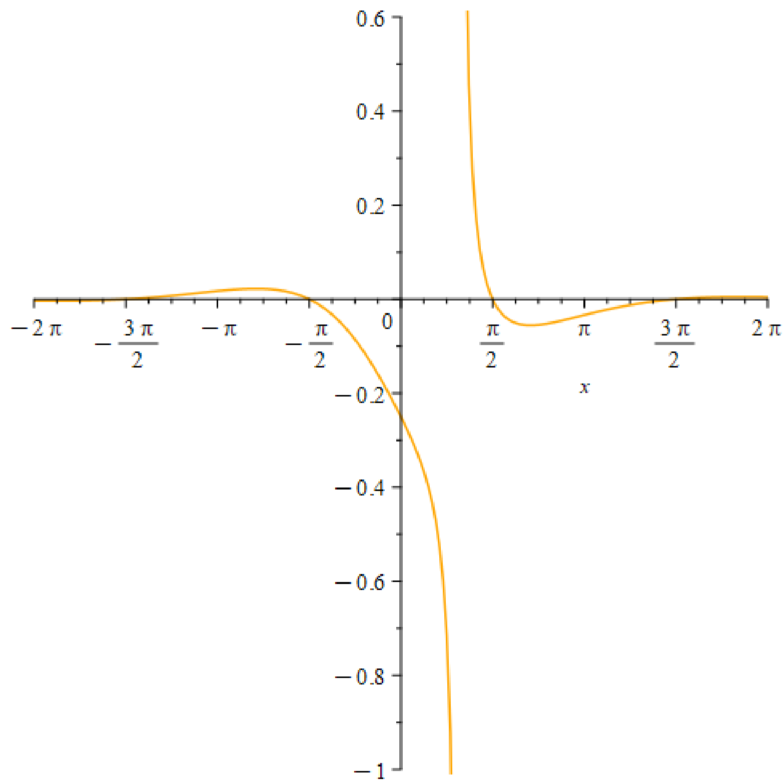

FIGURE 3. Plot of $\frac{\cos x}{(x^2+4)(x-1)}$.

$$= \Re \int_{-\infty}^{\infty} \frac{xe^{ix}}{(x-1)(x-2)} dx \tag{25}$$

There are poles at $x = 1, 2$ with residues

$$\begin{aligned} \text{Res}(1) &= -e^{i} \\ \text{Res}(2) &= 2e^{2i} \end{aligned} \tag{26}$$

There are no poles in the upper half plane, so the contour integral is

$$\lim_{r\to 0}\left[\int_{-\infty}^{r1-r} + \int_{S_1} + \int_{1+r}^{2-r} + \int_{S_2} + \int_{2+r}^{\infty}\right] \frac{xe^{ix}}{(x-1)(x-2)} dx = 0 \tag{27}$$

The two indentations give us

$$\left[\int_{S_1} + \int_{S_2}\right] \frac{xe^{ix}}{(x-1)(x-2)} dx = -i\pi\left(-e^{i} + 2e^{2i}\right) \tag{28}$$

so

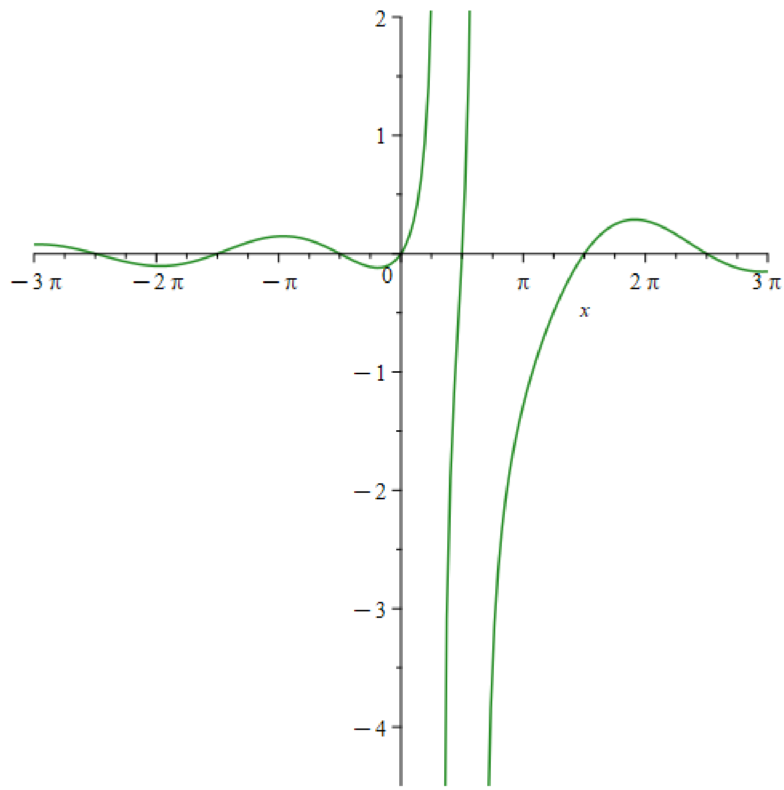

FIGURE 4. Plot of $\frac{x\cos x}{x^2-3x+2}$.

$$\int_{-\infty}^{\infty}\frac{xe^{ix}}{(x-1)(x-2)}dx = i\pi\left(-e^{i}+2e^{2i}\right) \tag{29}$$

$$= \pi(\sin 1-2\sin 2)+i\pi(-\cos 1+2\cos 2) \tag{30}$$

The real part gives us

$$I=\int_{-\infty}^{\infty}\frac{x\cos x}{x^2-3x+2}dx=\pi(\sin 1-2\sin 2)\approx -3.0697 \tag{31}$$

The bonus from the imaginary part is

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2-3x+2}dx=\pi(2\cos 2-\cos 1)\approx -4.3121 \tag{32}$$

Plots of the integrands in 31 and 32 are shown in 4 and 5.

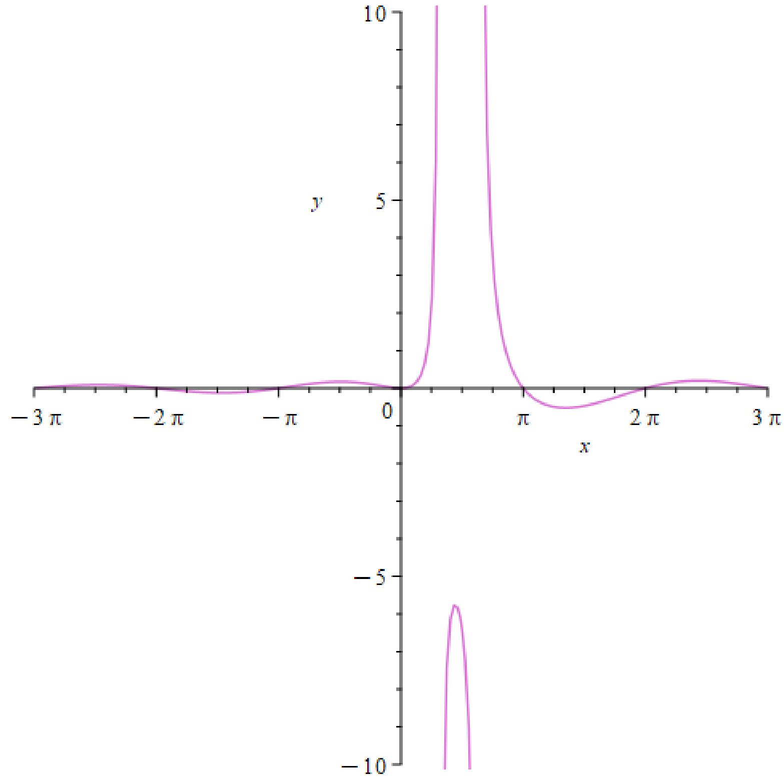

FIGURE 5. Plot of $\frac{x \sin x}{x^2-3x+2}$.