IMPROPER INTEGRALS WITH INDENTED CONTOURS

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Our earlier examples of improper integrals using Cauchy's residue theorem relied on poles not being on the real axis. Here we give examples of integrals of functions with poles on the real axis. The general technique is given in Saff and Snider, section 6.5. Suppose we have an integral

$$I=\int_{-\infty}^{\infty}f\left(x\right)dx \tag{1}$$

where f has a pole at $x=c$. We draw a contour lying along the real axis from $-\rho$ to ρ and closed by a semicircular arc in the upper half plane which we need to show becomes 0 in the limit $\rho\to\infty$. To cope with the pole, we draw a small semicircular arc S_r of radius r around $x=c$. The arc S_r runs clockwise around the pole, and lies in the upper half plane, so the overall contour excludes the pole, and thus the integral around this contour is zero. In the limit $\rho\to\infty$ we therefore have the integral

$$\left[\int_{-\infty}^{r}+\int_{S_r}+\int_{r}^{\infty}\right]f\left(z\right)dz \tag{2}$$

where $f\left(z\right)$ is $f\left(x\right)$ with the real variable x replaced by a complex variable z. We can then take the limit as $r\to0$ to get the real integral 1. We can use the lemma for evaluating integrals of circular arcs about poles to find the integral around S_r.

Example 1. Find

$$I=\int_{-\infty}^{\infty}\frac{e^{2ix}}{x+1}dx \tag{3}$$

We first observe that the limit of the integral around the large semicircular arc is zero by Jordan's lemma. There is a simple pole at $x=-1$, with residue

$$\text{Res}\left(-1\right)=e^{-2i} \tag{4}$$

The integral around the small clockwise arc around $x=-1$ has angle range π and is thus

$$\lim_{r\to 0}\int_{S_r}\frac{e^{2iz}}{z+1}dz=-i\pi e^{-2i} \tag{5}$$

From 2 we see that I is the negative of this, so

$$\int_{-\infty}^{\infty}\frac{e^{2ix}}{x+1}dx=i\pi e^{-2i} \tag{6}$$

Example 2. Find

$$I=\int_{-\infty}^{\infty}\frac{e^{ix}}{(x-1)(x-2)}dx \tag{7}$$

Here, we have two poles, at $x=1$ and $x=2$. The residues are

$$\text{Res}(1)=-e^{i} \tag{8}$$

$$\text{Res}(2)=e^{2i} \tag{9}$$

so we add the integrals around the arcs surrounding the poles

$$\lim_{r\to 0}\left(\int_{S_{r_1}}+\int_{S_{r_2}}\right)\frac{e^{iz}}{(z-1)(z-2)}dz=-i\pi\left(e^{2i}-e^{i}\right) \tag{10}$$

and

$$I=i\pi\left(e^{2i}-e^{i}\right) \tag{11}$$

Example 3. Find

$$I=\int_{0}^{\infty}\frac{\sin(2x)}{x(x^2+1)^2}dx \tag{12}$$

Since the integrand is an even function, we have

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(2x)}{x(x^2+1)^2}dx \tag{13}$$

We need to be careful with this integrand, as although it might appear to have a pole at $x=0$, in fact it has only a removable singularity here, because

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=2 \tag{14}$$

This means that trying to find the integral around a small semicircle about $z=0$ cannot be done using the residue theorem, as the residue at $x=0$ is 0. A better approach is to recognize that

$$I=\Im\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}dx\right] \tag{15}$$

Now there is a proper pole at $x=0$, and also another pole at $x=i$ in the upper half plane (the pole at $x=-i$ lies outside the contour). The residues are

$$\text{Res}\,(0)=\lim_{x\to0}\frac{e^{2ix}}{\left(x^{2}+1\right)^{2}}=1 \tag{16}$$

$$\text{Res}\,(i)=\lim_{x\to i}\frac{d}{dx}\left[(x-i)^{2}\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}\right] \tag{17}$$

$$=\lim_{x\to i}\frac{d}{dx}\left[\frac{e^{2ix}}{x\left(x+i\right)^{2}}\right] \tag{18}$$

$$=-e^{-2} \tag{19}$$

where I used Maple to get the last line. Therefore (as the integral over the outer semicircle goes to zero)

$$\left[\int_{-\infty}^{-r}+\int_{S_{r}}+\int_{r}^{\infty}\right]\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}dx=2\pi i\text{Res}\,(i)=-2\pi ie^{-2} \tag{20}$$

To get the integral $\int_{S_r}$ we use 16 to find

$$\lim_{r\to0}\int_{S_{r}}\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}dx=-i\pi\text{Res}\,(0)=-i\pi \tag{21}$$

Putting it all together, we have

$$\int_{-\infty}^{\infty}\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}dx=i\pi\left(1-2e^{-2}\right) \tag{22}$$

This is totally imaginary, so from 15 we have

$$\int_{0}^{\infty}\frac{\sin\left(2x\right)}{x\left(x^{2}+1\right)^{2}}dx=\frac{1}{2}\Im\int_{-\infty}^{\infty}\frac{e^{2ix}}{x\left(x^{2}+1\right)^{2}}dx \tag{23}$$

$$=\pi\left(\frac{1}{2}-e^{-2}\right) \tag{24}$$

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