IMPROPER INTEGRALS WITH TRIG FUNCTIONS AND COMPLEX POLYNOMIALS

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We've seen some examples of calculating improper integrals involving trig functions by taking the real or imaginary part of an integral with a complex exponential. Here, we'll look at an example where this technique doesn't work.

Example 1. Find

$$I=\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^{2}+4i}dx \tag{1}$$

Taking the real part of $\int_{-\infty}^{\infty}\frac{e^{3xi}}{x^{2}+4i}dx$ doesn't work here because of the $4i$ in the denominator. However, we can write

$$\cos(3x)=\frac{1}{2}\left(e^{3xi}+e^{-3xi}\right) \tag{2}$$

so

$$I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{3xi}}{x^{2}+4i}dx+\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-3xi}}{x^{2}+4i}dx\equiv I_{1}+I_{2} \tag{3}$$

We can evaluate I_1 and I_2 separately, using different contours. For I_1, we use the semicircular arc in the upper half plane, and for I_2, we use the lower half plane. Note, however, that the contour for I_2 is traversed clockwise in order that we travel left to right along the real axis.

The denominator has poles at

$$\sqrt{2}(1-i),\sqrt{2}(-1+i) \tag{4}$$

The first pole lies in the lower half plane, and the second in the upper half plane, so we need only one residue for each of I_1 and I_2. We have

$$I_{1}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{3xi}}{\left(x-\sqrt{2}(1-i)\right)\left(x-\sqrt{2}(-1+i)\right)}dx \tag{5}$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-3xi}}{\left(x-\sqrt{2}(1-i)\right)\left(x-\sqrt{2}(-1+i)\right)}dx \tag{6}$$

I used Maple to calculate the residues, which gets a bit messy if done by hand. We have

$$\text{Res}\left(\frac{1}{2}\frac{e^{3xi}}{\left(x-\sqrt{2}(1-i)\right)\left(x-\sqrt{2}(-1+i)\right)};\sqrt{2}(-1+i)\right) = -\frac{\sqrt{2}}{16}(1+i)e^{3\sqrt{2}(-1-i)} \tag{7}$$

$$\text{Res}\left(\frac{1}{2}\frac{e^{-3xi}}{\left(x-\sqrt{2}(1-i)\right)\left(x-\sqrt{2}(-1+i)\right)};\sqrt{2}(1-i)\right) = \frac{\sqrt{2}}{16}(1+i)e^{-3\sqrt{2}(1+i)} \tag{8}$$

To apply the residue theorem, we need to *subtract* the second residue from the first, since the second contour is traversed clockwise. Thus

$$I = 2\pi i\left[-\frac{\sqrt{2}}{16}(1+i)e^{3\sqrt{2}(-1-i)} - \frac{\sqrt{2}}{16}(1+i)e^{-3\sqrt{2}(1+i)}\right] \tag{9}$$

$$= \frac{\sqrt{2}\pi}{4}(1-i)e^{-3\sqrt{2}(1+i)} \tag{10}$$