IMPROPER INTEGRALS WITH TRIG FUNCTIONS

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We can extend the technique of finding improper integrals of rational functions to integrals of functions involving polynomials and trig functions. Integration of rational functions over the range $(-\infty,\infty)$ involved using a contour that extended left to right along the real axis and was closed by a semicircular arc in the positive imaginary half plane, then letting the contour expand to infinity. The success of the method relied on showing that the integral along the semicircular arc went to zero in the limit of its radius going to infinity.

We can use a similar argument to find integrals involving trigonometric functions. In what follows, we calculate residues using one of the formulas given here.

We use the definition of the complex exponential as

$$e^{iz}=\cos z+i\sin z \tag{1}$$

Example 1. Find

$$I=\int_{-\infty}^{\infty}\frac{\cos(2x)}{x^2+1}dx \tag{2}$$

First, we recognize that

$$I=\Re\int_{-\infty}^{\infty}\frac{e^{2iz}}{z^2+1}dz \tag{3}$$

In the upper half plane, z has a positive imaginary part, as in $z=x+iy$ with $y>0$. Thus the numerator becomes $e^{2ix}e^{-2y}$ and its magnitude goes to zero along the arc as the arc tends to infinity. [A detailed proof is given in Saff and Snider, section 6.4.] Therefore we have

$$I=\Re\left[2\pi i\sum_k r_k\right] \tag{4}$$

where r_k are the residues at the singularities of the integrand in the upper half plane. There is only one such singularity, at $z_0=i$, so we have

$$\text{Res}\left(\frac{e^{2iz}}{z^2+1};i\right) = -\frac{i}{2e^2} \tag{5}$$

so we have

$$I = \Re\left[2\pi i\left(-\frac{i}{2e^2}\right)\right] \tag{6}$$
$$= \Re\left(\frac{\pi}{e^2}\right) \tag{7}$$
$$= \frac{\pi}{e^2} \tag{8}$$

As a bonus, we note that since the quantity $\frac{\pi}{e^2}$ is already real, the imaginary past gives us

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2+1}dx = \Im\int_{-\infty}^{\infty} \frac{e^{2iz}}{z^2+1}dz = 0 \tag{9}$$

Actually, we could have deduced this without any calculations, since $\sin(2x)$ is an odd function and x^2+1 is an even function, so the integrand as a whole is odd, which gives an integral of 0 over any interval symmetric about the origin.

Example 2. Find

$$I = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10}dx \tag{10}$$
$$= \Im\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-2x+10}dx \tag{11}$$

Using the quadratic formula on the denominator, the integrand has poles at

$$x = 1\pm 3i \tag{12}$$

so the only pole in the upper half plane is at $z_0 = 1+3i$. This has residue

$$\text{Res}(1+3i) = \left(\frac{1}{2}-\frac{i}{6}\right)e^{-3+i} \tag{13}$$
$$= \frac{1}{e^3}\left(\frac{1}{2}-\frac{i}{6}\right)(\cos 1 + i\sin 1) \tag{14}$$

Thus

$$I = \Im\left[2\pi i \frac{1}{e^3}\left(\frac{1}{2} - \frac{i}{6}\right)(\cos 1 + i\sin 1)\right] \tag{15}$$

$$= \frac{\pi}{3e^3}(3\cos 1 + \sin 1) \tag{16}$$

Again, as a bonus, we can take the real part to get

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2 - 2x + 10} dx = \frac{\pi}{3e^3}(\cos 1 - 3\sin 1) \tag{17}$$

Example 3. Find

$$I = \int_0^{\infty} \frac{\cos x}{(x^2+1)^2} dx \tag{18}$$

We note the integrand is an even function, so

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx \tag{19}$$

$$= \frac{1}{2}\Re\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx \tag{20}$$

$$= \frac{1}{2}\Re\int_{-\infty}^{\infty} \frac{e^{ix}}{(x+i)^2 (x-i)^2} dx \tag{21}$$

This has a pole of order 2 in the upper half plane, at $z_0 = i$. The residue is

$$\text{Res}(i) = -\frac{i}{2e} \tag{22}$$

so

$$I = \frac{1}{2}\Re\left[2\pi i\left(-\frac{i}{2e}\right)\right] \tag{23}$$

$$= \frac{\pi}{2e} \tag{24}$$

Note that in this case, we can't get a bonus result by taking the imaginary part, since the original integral was over $(0,\infty)$, not $(-\infty,\infty)$. Thus we can say that

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx = 0 \tag{25}$$

since the integrand is odd, but there's no simple solution for

$$\int_0^\infty \frac{\sin x}{(x^2+1)^2} dx \tag{26}$$

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