INNER PRODUCTS AND HILBERT SPACES

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1. INNER PRODUCTS

An inner product defined on a vector space V is a function that maps each ordered pair of vectors (u, v) of V to a number denoted by $\langle u, v \rangle \in \mathbb{F}$ where $\mathbb{F}$ is either the set $\mathbb{R}$ of real numbers or the set $\mathbb{C}$ of complex numbers. The inner product satisfies the following axioms:

(1) positivity: $\langle v, v \rangle \geq 0$ for all $v \in V$. Note that $\langle v, v \rangle$ is always a *real* number.
(2) definiteness: $\langle v, v \rangle = 0$ if and only if $v = 0$ (the zero vector).
(3) additivity in the second slot: $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.
(4) homogeneity in the second slot: $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$ where $\lambda \in \mathbb{F}$.
(5) conjugate symmetry: $\langle u, v \rangle = \langle v, u \rangle^*$. Thus the inner product of two different vectors can be a complex number.

[Axler requires additivity and homogeneity in the first slot rather than the second, but Zwiebach uses the conditions above, which are more usual for physics.]

The *norm* $|v|$ of a vector v is defined as

norm of a vector

$$|v|^2 \equiv \langle v, v \rangle \tag{1}$$

All the above applies to both real and complex vector spaces, although it should be noted that in the case of conjugate symmetry in a real space, $\langle v, u \rangle^* = \langle v, u \rangle$ so in that case, the condition reduces to $\langle u, v \rangle = \langle v, u \rangle$.

Conditions 3 and 4 apply to the first slot as well, though with a slight difference for complex vector spaces. [These properties can actually be derived from the 5 axioms above, so aren't listed as separate axioms]:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \tag{2}$$
$$\langle \lambda u, v \rangle = \lambda^* \langle u, v \rangle \tag{3}$$

Two vectors are *orthogonal* if $\langle u, v \rangle = \langle v, u \rangle = 0$. By this definition, the zero vector is orthogonal to all vectors, including itself.

orthogonal vectors

The inner product is *non-degenerate*, meaning that any vector that is orthogonal to all vectors must be zero.

An *orthogonal decomposition* is defined for a vector u as follows: Suppose $u, v \in V$ and $v \neq 0$. We can write u as

orthogonal decomposition

$$u = \frac{\langle u, v \rangle}{|v|^2} v + u - \frac{\langle u, v \rangle}{|v|^2} v \tag{4}$$

$$\equiv cv + w \tag{5}$$

where

$$c = \frac{\langle u, v \rangle}{|v|^2} \in \mathbb{F} \tag{6}$$

$$w = u - \frac{\langle u, v \rangle}{|v|^2} v \tag{7}$$

From 1 and 2, $\langle w, v \rangle = 0$ so we've decomposed u into a component cv 'parallel' to v and another component w orthogonal to v.

There are a couple of important theorems for which we'll run through the proofs:

Theorem 1. *Pythagorean theorem. If u and v are orthogonal then*

$$|u + v|^2 = |u|^2 + |v|^2 \tag{8}$$

Proof. From the definition of the norm

$$|u + v|^2 = \langle u + v, u + v \rangle \tag{9}$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \tag{10}$$

$$= \langle u, u \rangle + \langle v, v \rangle \tag{11}$$

$$= |u|^2 + |v|^2 \tag{12}$$

where in the third line we used the orthogonality condition $\langle u, v \rangle = \langle v, u \rangle = 0$. $\square$

This is a generalization of the Pythagorean theorem for right-angled triangles, in which the two shorter sides are represented by the vectors u and v and the hypotenuse by the vector sum $u + v$. In this case, however, the vectors u and v can be any orthogonal vectors from any vector space, so the theorem applies to complex vector spaces as well.

Theorem 2. *Schwarz (or Cauchy-Schwarz) inequality. For all vectors $u, v \in V$*

$$|\langle u, v \rangle| \leq |u|\,|v| \tag{13}$$

Proof. The inequality is obviously true (as an equality) if $v = 0$, so we need to prove it for $v \neq 0$. In that case we can form an orthogonal decomposition of u as in 4 above:

$$u = \frac{\langle u, v \rangle}{|v|^2} v + w \tag{14}$$

Since v and w are orthogonal, we can apply the Pythagorean theorem:

$$
\begin{aligned}
|u|^2 &= \left| \frac{\langle u, v \rangle}{|v|^2} v \right|^2 + |w|^2 \tag{15} \\
&= \frac{|\langle u, v \rangle|^2}{|v|^4} |v|^2 + |w|^2 \tag{16} \\
&= \frac{|\langle u, v \rangle|^2}{|v|^2} + |w|^2 \tag{17} \\
&\geq \frac{|\langle u, v \rangle|^2}{|v|^2} \tag{18}
\end{aligned}
$$

Rearranging the last line and taking the positive square root (since norms are always non-negative) we have

$$|\langle u, v \rangle| \leq |u|\,|v| \tag{19}$$

$\square$

Theorem 3. *Triangle inequality. For all $u, v \in V$*

$$|u + v| \leq |u| + |v| \tag{20}$$

Proof. As with the Schwarz inequality, we start with the square of the norm:

$$
\begin{aligned}
|u + v|^2 &= \langle u + v, u + v \rangle \tag{21} \\
&= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle \tag{22} \\
&= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle u, v \rangle^* \tag{23} \\
&= |u|^2 + |v|^2 + 2\Re \langle u, v \rangle \tag{24} \\
&\leq |u|^2 + |v|^2 + 2|\langle u, v \rangle| \tag{25}
\end{aligned}
$$

The last line follows because $\langle u, v \rangle$ is a complex number, so

$$|\langle u,v \rangle| = \sqrt{(\mathfrak{R}\langle u,v \rangle)^2 + (\mathfrak{I}\langle u,v \rangle)^2} \geq \mathfrak{R}\langle u,v \rangle \tag{26}$$

We can now apply the Schwarz inequality 13 to the last term in the last line to get

$$
\begin{align}
|u+v|^2 &\leq |u|^2 + |v|^2 + 2|u||v| \tag{27} \\
&= (|u| + |v|)^2 \tag{28}
\end{align}
$$

Taking the positive square root, we get

$$|u+v| \leq |u| + |v| \tag{29}$$

$\square$

2. HILBERT SPACE

A finite-dimensional complex vector space with an inner product is a *Hilbert space.* An infinite-dimensional complex vector space with an inner product is also a Hilbert space if a completeness property holds. This property is a technical property which is always satisfied in quantum mechanics, so we can assume that any infinite-dimensional complex vector spaces we encounter in quantum theory are Hilbert spaces.

REFERENCES

(1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.
(2) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

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