INTEGRATION OF FRACTIONAL EXPONENTS 2

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Here are another couple of examples of integrating around a branch cut.

Example 1. Find

$$I=\int_0^{\infty}\frac{x^{\alpha}}{(x+9)^2}dx \tag{1}$$

where $-1<\alpha<1$. As before, we have a branch cut along the positive real axis, so the argument of $z=re^{i\theta}$ lies in the range $0<\theta<2\pi$. There is a pole of order 2 at $x=-9$. See Fig. 1

The contour integral is

$$\lim\left[\int_{\gamma_1}+\int_{C_\rho}+\int_{\gamma_2}+\int_{\Gamma_\varepsilon}\right]\frac{z^{\alpha}}{(z+9)^2}dz=2\pi i\text{Res}\left(z=-9\right) \tag{2}$$

where the limits are $\varepsilon\to 0$ and $\rho\to\infty$.

Along the top of the branch cut, we have

$$z=re^{i\theta} \tag{3}$$

while along the bottom, we have

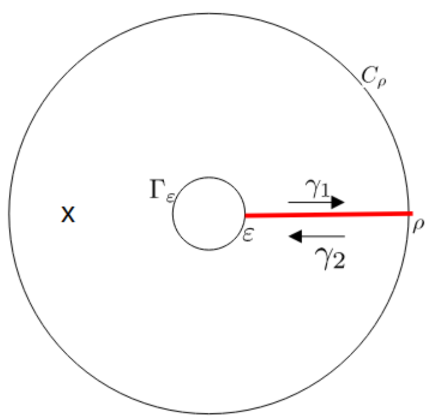

FIGURE 1. Contour for example 1. Pole of $z=-9$ at the point marked x.

$$z = re^{i(\theta+2\pi)} \tag{4}$$

Therefore

$$\lim \int_{\gamma_1} \frac{z^\alpha}{(z+9)^2} dz = \int_0^\infty \frac{x^\alpha}{(x+9)^2} dx \tag{5}$$

$$\lim \int_{\gamma_2} \frac{z^\alpha}{(z+9)^2} dz = \int_\infty^0 \frac{x^\alpha e^{2\pi\alpha i}}{(x+9)^2} dx \tag{6}$$

$$= -e^{2\pi\alpha i} e^{-2\pi i} \int_0^\infty \frac{x^\alpha}{(x+9)^2} dx \tag{7}$$

$$= -e^{2\pi\alpha i} \int_0^\infty \frac{x^\alpha}{(x+9)^2} dx \tag{8}$$

where the minus sign in the last two lines comes from reversing the direction of integration.

The residue is

$$\text{Res}(z=-9) = \lim_{z\to-9} \frac{d}{dx}\left[(x+9)^2 \frac{x^\alpha}{(x+9)^2}\right] \tag{9}$$

$$= \lim_{z\to-9} \alpha x^{\alpha-1} \tag{10}$$

$$= \alpha \frac{(-1)^\alpha}{-1} 9^{\alpha-1} \tag{11}$$

$$= -9^{\alpha-1} \alpha e^{i\pi\alpha} \tag{12}$$

Putting it all together, we have

$$\left(1-e^{2\pi\alpha i}\right) \int_0^\infty \frac{x^\alpha}{(x+9)^2} dx + \lim \left[\int_{C_\rho} + \int_{\Gamma_\varepsilon}\right] \frac{z^\alpha}{(z+9)^2} dz = 2\pi i \left(-9^{\alpha-1} \alpha e^{i\pi\alpha}\right) \tag{13}$$

Using the same techniques as in the earlier post we can show, using the fact that $-1 < \alpha < 1$, that the integrals around the two circles go to zero in the limits, so we have

$$\int_0^\infty \frac{x^\alpha}{(x+9)^2} dx = \frac{2\pi i \left(-9^{\alpha-1} \alpha e^{i\pi\alpha}\right)}{1-e^{2\pi\alpha i}} \tag{14}$$

$$= 9^{\alpha-1} \pi\alpha \frac{2i}{e^{\pi\alpha i} - e^{-\pi\alpha i}} \tag{15}$$

$$= \frac{9^{\alpha-1} \pi\alpha}{\sin(\pi\alpha)} \tag{16}$$

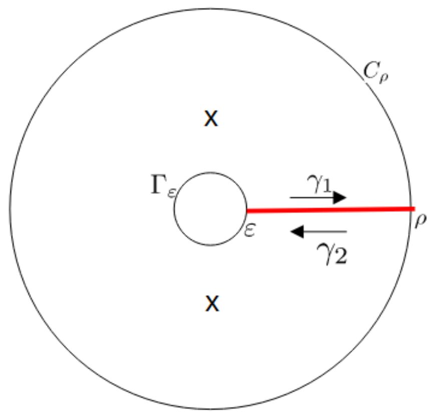

FIGURE 2. Contour for example 2. Poles at $z = \pm i$ at locations marked with x.

Example 2. Find

$$I = \int_0^\infty \frac{x^\alpha}{(x^2+1)^2} dx \tag{17}$$

We use the same branch cut as in the previous example. This time we have

$$I = \frac{x^\alpha}{(x+i)^2 (x-i)^2} dx \tag{18}$$

so there are order 2 poles at $z = \pm i$.

The contour integral is

$$\lim \left[\int_{\gamma_1} + \int_{C_\rho} + \int_{\gamma_2} + \int_{\Gamma_\varepsilon} \right] \frac{z^\alpha}{(z+i)^2 (z-i)^2} dz = 2\pi i \left(\text{Res}(i) + \text{Res}(-i) \right) \tag{19}$$

where the limits are $\varepsilon \to 0$ and $\rho \to \infty$.

Along the top of the branch cut, we have

$$z = re^{i\theta} \tag{20}$$

while along the bottom, we have

$$z = re^{i(\theta + 2\pi)} \tag{21}$$

Putting it all together, we have

$$\left(1-e^{2\pi\alpha i}\right)\int_0^\infty \frac{x^\alpha}{\left(x^2+1\right)^2}dx+\lim\left[\int_{C_\rho}+\int_{\Gamma_\varepsilon}\right]\frac{z^\alpha}{\left(z^2+1\right)^2}dz=2\pi i\left(\text{Res}\left(i\right)+\text{Res}\left(-i\right)\right) \tag{22}$$

We first find the residues. At $z=i$ we have

$$\text{Res}\left(i\right)=\lim_{z\to i}\frac{d}{dz}\left[\left(z-i\right)^2\frac{z^\alpha}{\left(z+i\right)^2\left(z-i\right)^2}\right] \tag{23}$$

$$=\lim_{z\to i}\frac{z^{\alpha-1}\left(\left(\alpha-2\right)z+i\alpha\right)}{\left(z+i\right)^3} \tag{24}$$

where I used Maple to simplify the derivative.

To ensure we use the correct branch, we must use the argument for z that lies within $0\le\theta\le2\pi$, so for $z\to i$ we use $z\to e^{i\pi/2}$ in the $z^{\alpha-1}$ factor. This gives us

$$\text{Res}\left(i\right)=-e^{i\pi\alpha/2}\frac{1-\alpha}{4}i \tag{25}$$

For the residue at $z=-i$ we must use $z=e^{3\pi i/2}$, so we have

$$\text{Res}\left(-i\right)=\lim_{z\to e^{3\pi i/2}}\frac{d}{dz}\left[\left(z+i\right)^2\frac{z^\alpha}{\left(z+i\right)^2\left(z-i\right)^2}\right] \tag{26}$$

$$=\lim_{z\to e^{3\pi i/2}}\frac{x^{\alpha-1}\left(\left(-\alpha+2\right)x+i\alpha\right)}{\left(-x+i\right)^3} \tag{27}$$

$$=e^{3\pi\alpha i/2}\frac{1-\alpha}{4}i \tag{28}$$

As before, we can show using earlier techniques that the integrals over the circles go to zero in the limits. We therefore have, adding 24 and 28:

$$\int_0^\infty\frac{x^\alpha}{\left(x^2+1\right)^2}dx=\frac{2\pi i\left(\text{Res}\left(i\right)+\text{Res}\left(-i\right)\right)}{1-e^{2\pi\alpha i}} \tag{29}$$

$$=\frac{2\pi i\left(1-\alpha\right)e^{\alpha\pi i/2}\left(e^{\pi\alpha i}-1\right)i}{4\left(1-e^{2\pi\alpha i}\right)} \tag{30}$$

$$=\frac{2\pi\left(1-\alpha\right)e^{\alpha\pi i/2}\left(1-e^{\pi\alpha i}\right)}{4\left(1-e^{\pi\alpha i}\right)\left(1+e^{\pi\alpha i}\right)} \tag{31}$$

$$=\frac{2\pi\left(1-\alpha\right)}{4\left(1+e^{\pi\alpha i}\right)}e^{\alpha\pi i/2} \tag{32}$$

$$= \frac{\pi(1-\alpha)}{4} \frac{2}{e^{-\pi\alpha i/2} + e^{\pi\alpha i/2}} \tag{33}$$

$$= \frac{\pi(1-\alpha)}{4\cos(\alpha\pi/2)} \tag{34}$$

In the third line, we used $i^2 = -1$ to reverse the sign in the $\left(e^{\pi\alpha i} - 1\right)$ factor. We also used $1 - e^{2\pi\alpha i} = \left(1 - e^{\pi\alpha i}\right)\left(1 + e^{\pi\alpha i}\right)$ (the difference of squares formula).