INTEGRATION OF FRACTIONAL EXPONENTS

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Here's another example of integrating around a branch cut.

Find

$$I=\int_{0}^{\infty}\frac{x^{\alpha-1}}{x+1}dx \tag{1}$$

where $0<\alpha<1$. As before, we have a branch cut along the positive real axis, so the argument of $z=re^{i\theta}$ lies in the range $0<\theta<2\pi$. There is also a simple pole at $x=-1$. See Fig. 1.

The contour integral is

$$\lim\left[\int_{\gamma_1}+\int_{C_\rho}+\int_{\gamma_2}+\int_{\Gamma_\varepsilon}\right]\frac{z^{\alpha-1}}{z+1}dz=2\pi i\text{Res}\left(z=-1\right) \tag{2}$$

where the limits are $\varepsilon\to 0$ and $\rho\to\infty$.

Along the top of the branch cut, we have

$$z=re^{i\theta} \tag{3}$$

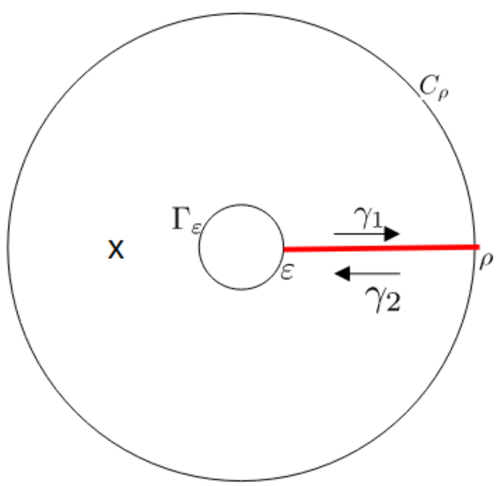

FIGURE 1. Contour of integration, with pole at x.

while along the bottom, we have

$$z = re^{i(\theta+2\pi)} \tag{4}$$

Therefore

$$\lim \int_{\gamma_1} \frac{z^{\alpha-1}}{z+1} dz = \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx \tag{5}$$

$$\lim \int_{\gamma_2} \frac{z^{\alpha-1}}{z+1} dz = \int_\infty^0 \frac{x^{\alpha-1} e^{2\pi(\alpha-1)i}}{x+1} dx \tag{6}$$

$$= -e^{2\pi\alpha i} e^{-2\pi i} \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx \tag{7}$$

$$= -e^{2\pi\alpha i} \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx \tag{8}$$

where the minus sign in the last two lines comes from reversing the direction of integration.

For the residue, we have, using $-1 = e^{i\pi}$

$$\text{Res}(-1) = \lim_{z\to-1} \left(z^{\alpha-1}\right) \tag{9}$$

$$= e^{i\pi(\alpha-1)} \tag{10}$$

$$= e^{i\pi\alpha} e^{-i\pi} \tag{11}$$

$$= -e^{i\pi\alpha} \tag{12}$$

Putting it together, we have

$$\left(1 - e^{2\pi\alpha i}\right) \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx + \lim \left[\int_{C_\rho} + \int_{\Gamma_\varepsilon}\right] \frac{z^{\alpha-1}}{z+1} dz = 2\pi i \left(-e^{i\pi\alpha}\right) \tag{13}$$

It remains to show that the integrals around the circles C_ρ and Γ_ε go to zero in the limits. For C_ρ, we have

$$\left|\frac{z^{\alpha-1}}{z+1}\right| \le \frac{\rho^{\alpha-1}}{\rho-1} \tag{14}$$

$$\int_{C_\rho} \frac{z^{\alpha-1}}{z+1} dz \le 2\pi\rho \frac{\rho^{\alpha-1}}{\rho-1} = 2\pi \frac{\rho^\alpha}{\rho-1} \to 2\pi\rho^{\alpha-1} \tag{15}$$

Since $0 < \alpha < 1$, the exponent $\alpha - 1$ lies in the range $(-1, 0)$, so $\rho^{\alpha-1} \to 0$ as $\rho \to \infty$.

For Γ_ε we have

$$\left|\frac{z^{\alpha-1}}{z+1}\right| \leq \frac{\varepsilon^{\alpha-1}}{1-\varepsilon} \tag{16}$$

$$\int_{\Gamma_\varepsilon} \frac{z^{\alpha-1}}{z+1} dz \leq 2\pi\varepsilon\frac{\varepsilon^{\alpha-1}}{1-\varepsilon} = 2\pi\frac{\varepsilon^{\alpha}}{1-\varepsilon} \tag{17}$$

This also goes to zero in the limit $\varepsilon \to 0$.

Therefore from 13

$$\left(1-e^{2\pi\alpha i}\right)\int_0^\infty \frac{x^{\alpha-1}}{x+1} dx = -2\pi i e^{i\pi\alpha} \tag{18}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{x+1} dx = \pi\frac{e^{i\pi\alpha}2i}{e^{2\pi\alpha i}-1} \tag{19}$$

$$= \pi\frac{2i}{e^{i\pi\alpha}-e^{-i\pi\alpha}} \tag{20}$$

$$= \frac{\pi}{\sin\left(\pi\alpha\right)} \tag{21}$$

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