INTEGRATION WITH BRANCH CUTS

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A more involved form of contour integration occurs when the integrand contains a multiple-valued function, such as $\sqrt{z}$. Such functions can be defined over two or more branches with a branch point and branch cut being used to define the edges of a particular branch. The process is best illustrated with an example.

We want to find

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1} dx \tag{1}$$

The square root has two possible values for a given complex number z. If we write

$$z = e^{\text{Log}r + i\theta} \tag{2}$$

then the two values are

$$\sqrt{z} = \begin{cases} e^{(\text{Log}r)/2} e^{i\theta/2} \\ e^{(\text{Log}r)/2} e^{i(\theta+2\pi)/2} \end{cases} \tag{3}$$

We can take the branch point to be at $z = 0$ and the branch cut to be the positive real axis. We are thus faced with the integral around the contour shown in Fig. 1.

The integral proceeds outwards along the upper side of the branch cut γ_1 from the circle Γ_ε of radius ε around the branch point at $z = 0$, out to the point $z = \rho$ on the outer circle C_ρ. We then integrate counterclockwise around C_ρ, then backwards around the lower side of the branch cut γ_2. We then need to take the limits as $\varepsilon \to 0$ and $\rho \to \infty$. If the integrals around the two circles go to zero in the limit, then the integral along γ_1 will be the integral we want, while the integral along γ_2 (since it is taken in the opposite direction) will also be the required integral, as we'll see below.

Saff and Snider show in Section 6.6 Example 1 that this contour integral is

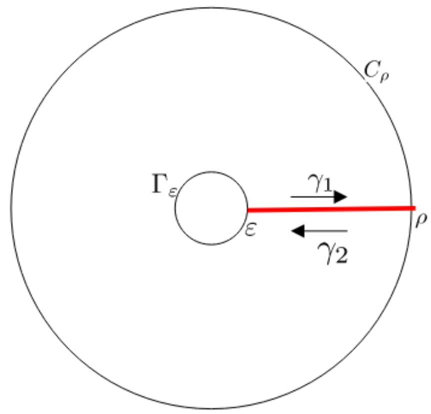

FIGURE 1. Contour for integration around a branch cut.

$$\left[\int_{\gamma_1}+\int_{C\rho}+\int_{\gamma_2}+\int_{\Gamma_\varepsilon}\right]=2\pi i\left(\sum \text{residues inside the contour}\right) \tag{4}$$

As the inner circle shrinks to zero and the outer circle expands to infinity, the contour will ultimately contain all poles of the integrand. For 1, there are simple poles at $z=\pm i$, so we need to find the residues there. We need to be careful to use the correct branch of the square root in doing this.

For $z=i=e^{i\pi/2}$ we have

$$\begin{aligned}
\operatorname{Res}(i) &= \lim_{z\to i}\left[(z-i)\frac{\sqrt{z}}{(z+i)(z-i)}\right] && (5)\\
&= \frac{e^{(i\pi/2)/2}}{i+i} && (6)\\
&= \frac{1}{2i}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) && (7)
\end{aligned}$$

For $z=-i=e^{3i\pi/2}$ we have

$$\begin{aligned}
\operatorname{Res}(-i) &= \lim_{z\to -i}\left[(z+i)\frac{\sqrt{z}}{(z+i)(z-i)}\right] && (8)\\
&= \frac{e^{(3i\pi/2)/2}}{-i-i} && (9)\\
&= -\frac{1}{2i}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) && (10)
\end{aligned}$$

$$= \frac{1}{2i}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \tag{11}$$

We therefore have

$$2\pi i\left[\text{Res}\,(i) + \text{Res}\,(-i)\right] = \pi\sqrt{2} \tag{12}$$

In the limit, γ_1 tends to the desired integral 1. For γ_2 we are integrating along the lower side of the branch cut, where $z = xe^{2\pi i}$, so

$$\sqrt{z} = \sqrt{x}e^{\pi i} = -\sqrt{x} \tag{13}$$

Thus

$$\int_{\gamma_2} \frac{\sqrt{z}}{z^2+1}dz = -\int_{\varepsilon}^{\rho} \frac{-\sqrt{x}}{x^2+1}dx \tag{14}$$

where the minus sign outside the integral on the RHS is due to the fact that the integral along γ_2 is actually from ρ to ε. Thus the integral along γ_2 is, in the limit, also equal to I.

All that remains is to show that the integrals around C_ρ and Γ_ε go to zero in the limit. We can do this by considering magnitudes of the integrand.

For Γ_ε we have

$$\left|\frac{\sqrt{z}}{z^2+1}\right| \leq \frac{\sqrt{\varepsilon}}{1-\varepsilon^2} \tag{15}$$

This follows because the smallest z^2+1 can be on the circle is when $z = \varepsilon i$. This gives the largest value of $1/\left(1-\varepsilon^2\right)$.

The circumference of Γ_ε is $2\pi\varepsilon$ so we have for $\varepsilon \to 0$

$$\int_{\Gamma_\varepsilon} \frac{\sqrt{z}}{z^2+1}dz \leq \frac{2\pi\varepsilon\sqrt{\varepsilon}}{1-\varepsilon^2} \to 0 \tag{16}$$

For C_ρ we have for $\rho \to \infty$

$$\left|\frac{\sqrt{z}}{z^2+1}\right| \leq \frac{\sqrt{\rho}}{\rho^2-1} \tag{17}$$

so

$$\int_{C_\rho} \frac{\sqrt{z}}{z^2+1}dz \leq \frac{2\pi\rho\sqrt{\rho}}{\rho^2-1} \to \frac{2\pi\rho^{3/2}}{\rho^2} \to 0 \tag{18}$$

We therefore have the final result by adding the integrals over γ_1 and γ_2:

$$\lim\left[\int_{\gamma_1} + \int_{\gamma_2}\right] \frac{\sqrt{x}}{x^2+1} = \sqrt{2}\pi = 2I \tag{19}$$

$$I=\int_{0}^{\infty}\frac{\sqrt{x}}{x^{2}+1}dx=\frac{\pi}{\sqrt{2}} \tag{20}$$

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