INVARIANCE OF LAPLACE'S EQUATION UNDER CONFORMAL MAPPING

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Suppose we have an analytic function $f(z)$ that acts on a domain D. The function maps each point in D onto another point in a domain D'. If the mapping is one-to-one, that is, for each point in D there is a unique point w in D', then the function $f(z)$ has an inverse $f^{-1}(w)$ that maps each point in D' back to a unique point in D. Such a mapping (and its inverse) is called a *conformal mapping*. We can think of a conformal mapping in geometric terms, by plotting D as a region in the complex plane, and then D' as another, possible different, region in the complex plane.

We can write a point in D as

$$z = x + iy \tag{1}$$

and a point in D' as

$$w = u + iv \tag{2}$$

If we start with a function $\phi(x,y)$ that satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{3}$$

in a domain D, then when we do a conformal mapping to D', the corresponding function $\psi(u,v)$ in D' also satisfies Laplace's equation (proof given in Saff and Snider, Section 7.1). That is,

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} = 0 \tag{4}$$

Example 1. Consider the mapping given by

$$f(z) = e^z \tag{5}$$

defined over the domain D given by the half-infinite strip bounded by the imaginary axis ($x = 0$) on the left, the horizontal line $y = \frac{\pi}{2}$ above, and the horizontal line $y = -\frac{\pi}{2}$ below. The strip extends off to infinity on the right. To find D' (where points are labelled by (u,v)), we see how the boundaries transform under the mapping 5. On the imaginary axis, $f(z)=e^{iy}$ which has modulus 1, so it maps the imaginary axis onto the unit circle. For $y=\frac{\pi}{2}$, $f(z)=e^{x}e^{i\pi/2}=ie^{x}$. Since $x>0$ in D, this maps to the imaginary axis with $v>1$. For $y=-\frac{\pi}{2}$, we have $f(z)=e^{x}e^{-i\pi/2}=-ie^{x}$, so this line is mapped into the negative imaginary axis with $v<-1$. A general point $f(z)=e^{x+iy}=e^{x}(\cos y+i\sin y)$ with $-\frac{\pi}{2}<y<\frac{\pi}{2}$ and $x>0$ thus maps to the right half-plane, with the semicircular disk $|w|<1$ excluded.

Now consider the harmonic function

$$\phi(x,y)=x+y \tag{6}$$

Since

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial y^2}=0 \tag{7}$$

this function satisfies Laplace's equation 3. To find its transform under $f(z)=e^{z}$, we have

$$\psi(u,v)=\phi(x(u,v),y(u,v)) \tag{8}$$

We have

$$w=f(z)=e^{x}\cos y+ie^{x}\sin y \tag{9}$$

so

$$\begin{aligned} u&=e^{x}\cos y\\ v&=e^{x}\sin y \end{aligned} \tag{10}$$

We can invert this to give

$$u^2+v^2=e^{2x} \tag{11}$$

$$x(u,v)=\frac{1}{2}\mathrm{Log}\left(u^2+v^2\right) \tag{12}$$

$$\frac{v}{u}=\tan y \tag{13}$$

$$y(u,v)=\arctan\frac{v}{u} \tag{14}$$

Therefore

$$\psi(u,v)=x(u,v)+y(u,v) \tag{15}$$

$$=\frac{1}{2}\mathrm{Log}\left(u^2+v^2\right)+\arctan\frac{v}{u} \tag{16}$$

We can verify that $\psi(u,v)$ satisfies Laplace's equation 4 by direct differentiation. The derivatives get a bit messy, so I used Maple to simplify the results. We get

$$\frac{\partial^2\psi}{\partial u^2} = \frac{-u^2+2uv+v^2}{\left(u^2+v^2\right)^2} \tag{17}$$

$$\frac{\partial^2\psi}{\partial v^2} = \frac{u^2-2uv-v^2}{\left(u^2+v^2\right)^2} \tag{18}$$

From this, we see that 4 is indeed satisfied.

If we start with

$$\psi(w) = u+v \tag{19}$$

then we can get the inverse relation $\phi(x,y)$ from

$$\phi(x,y) = \psi(u(x,y),v(x,y)) \tag{20}$$

Using 10 we have

$$\phi(x,y) = e^x\cos y + e^x\sin y \tag{21}$$

Calculating the derivatives, we have

$$\frac{\partial^2\phi}{\partial x^2} = e^x\cos y + e^x\sin y \tag{22}$$

$$\frac{\partial^2\phi}{\partial y^2} = -e^x\cos y - e^x\sin y \tag{23}$$

We see again, that Laplace's equation is satisfied, that is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{24}$$

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