INVERSE SECANT

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Post date: 9 January 2025.

Using a similar method to that for finding the inverse sine and cosine we can find the inverse secant. The secant is defined as the reciprocal of the cosine, so we have

$$\sec z = \frac{2}{e^{iz}+e^{-iz}} \tag{1}$$

Let

$$w = \sec^{-1} z \tag{2}$$

Then

$$z = \sec w = \frac{2}{e^{iw}+e^{-iw}} \tag{3}$$

Multiplying through by $\left(e^{iw}+e^{-iw}\right)/z$ we have

$$e^{iw}+e^{-iw} = \frac{2}{z} \tag{4}$$

Multiplying by e^{iw} and collecting terms gives

$$e^{2iw} - 2\frac{e^{iw}}{z} + 1 = 0 \tag{5}$$

Using the quadratic formula gives

$$e^{iw} = \frac{2/z + \sqrt{4/z^2 - 4}}{2} \tag{6}$$

$$= \frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2} \tag{7}$$

where we've taken the positive square root. Taking logs gives

$$w = \sec^{-1} z = -i\log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right] \tag{8}$$

We can compare the values given by this formula with those quoted in mathematical tables. First, consider $z = x > 1$. We can rewrite 8 as

$$\sec^{-1} x = -i\text{Log}\left[\frac{1}{x} + i\left(1 - \frac{1}{x^2}\right)^{1/2}\right] \tag{9}$$

where the quantity $1 - 1/x^2 > 0$ for $x > 1$ or $x < -1$, and we've taken the principal value of the logarithm Log z, which has an argument in the interval $(-\pi, \pi]$ As $x \to 1$ from above, the value of the Log tends to 0 from above, so

$$\sec^{-1} x \to -i\text{Log } 1 = 0 \tag{10}$$

As $x \to +\infty$, we have

$$\sec^{-1} x \to -i\text{Log } i = -i\left(i\frac{\pi}{2}\right) = \frac{\pi}{2} \tag{11}$$

Thus for $x \in (1, \infty)$, we have $\sec^{-1} x \in \left(0, \frac{\pi}{2}\right)$, which agrees with the standard values for $\sec^{-1} x$.

For $x < -1$, we look first at $x \to -1$ from below. In this case

$$\sec^{-1} x \to -i\text{Log } (-1) = -i\,(i\pi) = \pi \tag{12}$$

As $x \to -\infty$, we have

$$\sec^{-1} x \to -i\text{Log } i = \frac{\pi}{2} \tag{13}$$

Thus for $x \in (-\infty, -1)$, we have $\sec^{-1} x \in \left(\frac{\pi}{2}, \pi\right)$. Note that in Saff and Snider's book they state that the tabulated values in this case are $\sec^{-1} \in \left(-\pi, -\frac{\pi}{2}\right)$. However, a Google search for 'principal value of arcsec' (and Wikipedia) state that the values are actually $\sec^{-1} x \in \left(\frac{\pi}{2}, \pi\right)$ as we've found here.