INVERSE TRIGONOMETRIC FUNCTIONS

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The complex trigonometric functions are defined in terms of the exponential by

$$\begin{aligned} \cos z &\equiv \frac{e^{iz}+e^{-iz}}{2} \\ \sin z &\equiv \frac{e^{iz}-e^{-iz}}{2i} \end{aligned} \tag{1}$$

We can define the inverse trig functions using these definitions. We have

$$w = \arccos z = \cos^{-1} z \tag{2}$$

so

$$z = \cos w = \frac{e^{iw}+e^{-iw}}{2} \tag{3}$$

Multiplying through by $2e^{iw}$ gives

$$e^{2iw} - 2ze^{iw} + 1 = 0 \tag{4}$$

Using the quadratic equation we have

$$e^{iw} = \frac{2z+\sqrt{4z^2-4}}{2} \tag{5}$$

$$= z + \sqrt{z^2-1} \tag{6}$$

where the square root has two possible values. Taking logs, we have

$$w = \arccos z = -i\log\left(z + \left(z^2-1\right)^{1/2}\right) \tag{7}$$

The inverse sine is found by a similar calculation, giving

$$\arcsin z = -i\log\left(iz + \left(1-z^2\right)^{1/2}\right) \tag{8}$$

Example 1. Find z such that $\sin z=2$. From 8 we have

$$z=\arcsin 2=-i\log\left(2i+(-3)^{1/2}\right) \tag{9}$$
$$=-i\log\left(\left(2\pm\sqrt{3}\right)i\right) \tag{10}$$

The logarithm is

$$\log\left(\left(2\pm\sqrt{3}\right)i\right)=\text{Log}\left|2\pm\sqrt{3}\right|+i\arg\left[\left(2\pm\sqrt{3}\right)i\right] \tag{11}$$
$$=\text{Log}\left|2\pm\sqrt{3}\right|+i\left(\frac{\pi}{2}+2k\pi\right) \tag{12}$$

where the argument of $\left(2\pm\sqrt{3}\right)i$ is $\frac{\pi}{2}$ for both values, since $2-\sqrt{3}>0$. Therefore

$$z=-i\left[\text{Log}\left|2\pm\sqrt{3}\right|+i\left(\frac{\pi}{2}+2k\pi\right)\right] \tag{13}$$
$$=\frac{\pi}{2}+2k\pi-i\text{Log}\left|2\pm\sqrt{3}\right| \tag{14}$$

Note that since $\text{Log}\,\frac{1}{z}=-\text{Log}\,z$ and

$$\frac{1}{2-\sqrt{3}}=\frac{2+\sqrt{3}}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}=\frac{2+\sqrt{3}}{4-3}=2+\sqrt{3} \tag{15}$$

we can also write 14 as

$$z=\arcsin 2=\frac{\pi}{2}+2k\pi\pm i\text{Log}\left(2+\sqrt{3}\right) \tag{16}$$

Example 2. Find $z=\arccos 2i$. From 7 we have

$$z=-i\log\left(2i+(-4-1)^{1/2}\right) \tag{17}$$
$$=-i\log\left(\left(2\pm\sqrt{5}\right)i\right) \tag{18}$$
$$=-i\left[\text{Log}\left|2\pm\sqrt{5}\right|+i\arg\left(\left(2\pm\sqrt{5}\right)i\right)\right] \tag{19}$$

This time, $\arg\left(2+\sqrt{5}\right)i=\frac{\pi}{2}$, but $\arg\left(2-\sqrt{5}\right)i=-\frac{\pi}{2}$. There are thus two sets of solutions:

$$z=\begin{cases}\frac{\pi}{2}+2k\pi-i\text{Log}\left(\sqrt{5}+2\right)\\-\frac{\pi}{2}+2k\pi-i\text{Log}\left(\sqrt{5}-2\right)\end{cases} \tag{20}$$

where in the last line, we used $\left|2-\sqrt{5}\right|=\sqrt{5}-2$.

Again, we can rewrite this using

$$\frac{1}{\sqrt{5}-2}=\frac{\sqrt{5}+2}{\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)}=\sqrt{5}+2 \tag{21}$$

We get

$$z=\begin{cases}\frac{\pi}{2}+2k\pi-i\mathrm{Log}\left(\sqrt{5}+2\right)\\ -\frac{\pi}{2}+2k\pi+i\mathrm{Log}\left(\sqrt{5}+2\right)\end{cases} \tag{22}$$

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