INVERSION MAPPING

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The complex function

$$f(z)=\frac{1}{z} \tag{1}$$

is known as an *inversion mapping*. Its domain of definition is all z except for $z=0$. Some examples of regions of the complex plane that are transformed by an inversion mapping follow.

Example 1. Map the circle $|z|=r$. This is easiest if we represent z in polar form as

$$z=re^{i\theta} \tag{2}$$

Then

$$w=\frac{1}{z}=\frac{1}{r}e^{-i\theta} \tag{3}$$

The argument θ can take all values from $-\pi$ to π and r is a constant, so the mapping is to the circle with radius $\frac{1}{r}$, or $|w|=\frac{1}{r}$.

Example 2. Map the ray Arg $z=\theta_0$, where θ_0 is a constant in the interval $(-\pi,\pi)$. Again using polar form

$$z=re^{i\theta_0} \tag{4}$$

where now r can vary from 0 to ∞. We have

$$w=\frac{1}{z}=\frac{1}{r}e^{-i\theta_0} \tag{5}$$

As $\frac{1}{r}$ can take on all positive values, w is the ray with Arg $w=-\theta_0$.

Example 3. Map the circle $|z-1|=1$. First, we work out the LHS as

$$1^2 = |z-1|^2 = (x-1)^2 + y^2 \tag{6}$$
$$= x^2 + y^2 - 2x + 1 \tag{7}$$
$$= |z|^2 - 2x + 1 \tag{8}$$
$$= 1 \tag{9}$$

so

$$|z|^2 = 2x \tag{10}$$

Doing the mapping, we have

$$w = \left| \frac{1}{z} - 1 \right| = 1 \tag{11}$$

We have

$$\left| \frac{1}{z} - 1 \right| = \left| \frac{1-z}{z} \right| \tag{12}$$
$$= \frac{|z-1|}{|z|} = 1 \tag{13}$$

Squaring both sides we have

$$\frac{|z-1|^2}{|z|^2} = 1 \tag{14}$$

But we know that $|z-1| = 1$ and from 10 we have $|z|^2 = 2x$, so

$$\frac{1}{2x} = 1 \tag{15}$$

or

$$x = \frac{1}{2} \tag{16}$$

Thus the circle $|z-1| = 1$ maps to the vertical line $x = \frac{1}{2}$.

Example 4. Map a point (x_1, x_2, x_3) on the Riemann sphere under inversion. The original projection maps z onto (x_1, x_2, x_3), and is given by the relations

$$\begin{aligned} x_1 &= \frac{2\Re z}{|z|^2+1} \\ x_2 &= \frac{2\Im z}{|z|^2+1} \\ x_3 &= \frac{|z|^2-1}{|z|^2+1} \end{aligned} \tag{17}$$

Under inversion, we have

$$\begin{aligned} x_1' &= \frac{2\Re(1/z)}{|1/z|^2+1} \\ x_2' &= \frac{2\Im(1/z)}{|1/z|^2+1} \\ x_3' &= \frac{|1/z|^2-1}{|1/z|^2+1} \end{aligned} \tag{18}$$

The real and imaginary parts are

$$\begin{aligned} \Re\frac{1}{z} &= \frac{\Re\bar{z}}{|z|^2} = \frac{\Re z}{|z|^2} \\ \Im\frac{1}{z} &= \frac{\Im\bar{z}}{|z|^2} = -\frac{\Im z}{|z|^2} \end{aligned} \tag{19}$$

Substituting into 18 and multiplying top and bottom by $|z|^2$ gives

$$\begin{aligned} x_1' &= \frac{2\Re z}{1+|z|^2} = x_1 \\ x_2' &= -\frac{2\Im z}{1+|z|^2} = -x_2 \\ x_3' &= \frac{1-|z|^2}{1+|z|^2} = -x_3 \end{aligned} \tag{20}$$

Thus inversion results in a rotation by π about the x_1 axis of the sphere.

This has the interesting consequence that inversion maps any circle in the z plane to either a circle or a line in the w plane. To see this, recall that any circle in the z plane projects to a circle on the Riemann sphere. Thus this projection is rotated by π on the Riemann sphere under inversion and therefore remains as a circle on the sphere. When projected back onto the w plane, we again get either a circle or a line. Special cases of this are given in Example 1 (above; circle mapping to a circle) and Example 3 (circle mapping to a line).

PINGBACKS

Pingback: Joukowski mapping