JOUKOWSKI MAPPING

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The Joukowski mapping is defined by

$$w=J(z)=\frac{1}{2}\left(z+\frac{1}{z}\right) \tag{1}$$

If we apply the inversion mapping we get the same mapping back again:

$$J\left(\frac{1}{z}\right)=\frac{1}{2}\left(\frac{1}{z}+z\right)=J(z) \tag{2}$$

The Joukowski mapping applied to the unit circle gives the following. For this, it's easiest to express z in polar form

$$z=re^{i\theta} \tag{3}$$

The unit circle has $r=1$ so we get from 1

$$J\left(e^{i\theta}\right)=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \tag{4}$$

$$=\cos\theta \tag{5}$$

Over the unit circle, $\cos\theta$ varies from -1 to 1, so the mapping is to the line segment $[-1,1]$ on the real (x) axis.

We can generalize this to find the mapping for a circle $|z|=r$. In this case, we have

$$J\left(re^{i\theta}\right)=\frac{1}{2}\left(re^{i\theta}+\frac{1}{r}e^{-i\theta}\right) \tag{6}$$

$$=\frac{1}{2}\left[\left(r+\frac{1}{r}\right)\cos\theta+i\left(r-\frac{1}{r}\right)\sin\theta\right] \tag{7}$$

If we write $w=J(z)=u+iv$, we have

$$\begin{aligned} u &= \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta \\ v &= \frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta \end{aligned} \tag{8}$$

Rearranging, we get

$$\begin{aligned} \cos\theta &= \frac{u}{\frac{1}{2}\left(r+\frac{1}{r}\right)} \\ \sin\theta &= \frac{v}{\frac{1}{2}\left(r-\frac{1}{r}\right)} \end{aligned} \tag{9}$$

Therefore

$$\frac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2}+\frac{v^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2}=1 \tag{10}$$

For any given circle, r is a constant, so 10 is the equation of an ellipse in the standard form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{11}$$

The semi-major axis is

$$a=\frac{1}{2}\left(r+\frac{1}{r}\right) \tag{12}$$

and the semi-minor axis is

$$b=\frac{1}{2}\left(r-\frac{1}{r}\right) \tag{13}$$

The foci are located at $\pm c$ where

$$c^2 = a^2-b^2 \tag{14}$$

$$= \left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2-\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2 \tag{15}$$

$$= 1 \tag{16}$$

where I used Maple to do the algebra. Thus the foci are at $(\pm 1, 0)$ for any value of r.

The eccentricity E (I'm using an uppercase E so as not to confuse things with $e = 2.718\ldots$) is given by

$$E = \frac{c}{a} \tag{17}$$

$$= \frac{2}{r + \frac{1}{r}} \tag{18}$$

Although we must exclude $r = 1$ (due to division by 0 in 10), we see that the limiting case $r \to 1$ gives the line segment $[-1, 1]$ found above. This also gives the maximum eccentricity of 1.