JUMPING FROG PROBLEM

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Suppose there is a frog that starts at the origin of the complex plane, then jumps to the point $x=1$, then jumps a distance of $\frac{1}{2}$ at an angle α to the left, then $\frac{1}{4}$ at an angle α to the left of its previous path and so on to infinity. (We're ignoring practical constraints such as how a frog can jump tinier and tinier distances, but never mind.) The problem is to show that, no matter what angle α the frog chooses, after completing its journey it will have travelled a net distance d, and it will always end up at some point on the circle

$$\left|d-\frac{4}{3}\right|=\frac{2}{3} \tag{1}$$

First, we need to write down the series of jumps. A change of direction at an angle α to the left is represented by multiplying by $e^{i\alpha}$. Thus the jumps taken by the frog are

$$d=1+\frac{1}{2}e^{i\alpha}+\frac{1}{4}e^{2i\alpha}+\ldots=\sum_{j=0}^{\infty}\frac{e^{ij\alpha}}{2^{j}} \tag{2}$$

This is a geometric series of the form

$$d=\sum_{j=0}^{\infty}\left(\frac{e^{i\alpha}}{2}\right)^{j} \tag{3}$$

where the modulus of the base term is

$$\left|\frac{e^{i\alpha}}{2}\right|=\frac{1}{2}<1 \tag{4}$$

so the series converges. The sum is

$$d=\sum_{j=0}^{\infty}\left(\frac{e^{i\alpha}}{2}\right)^{j}=\frac{1}{1-e^{i\alpha}/2} \tag{5}$$

We can rationalize this by multiplying top and bottom by $1-e^{-i\alpha}/2$ to get

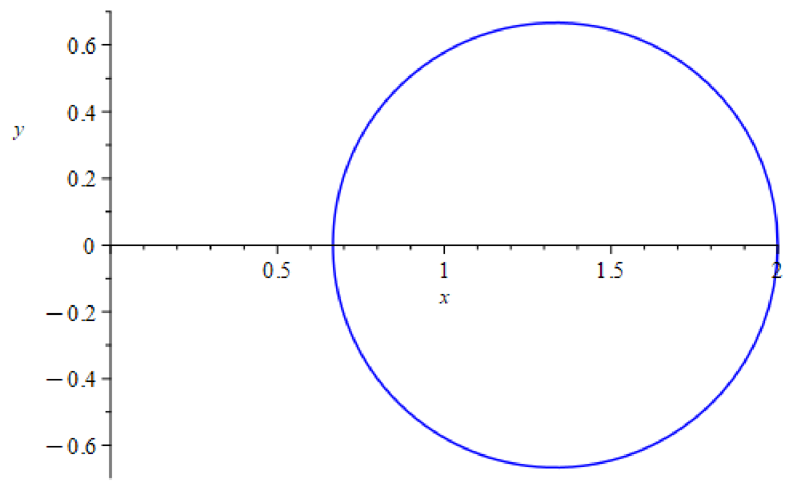

FIGURE 1. End points of frog's journey.

$$d = \frac{1 - e^{-i\alpha}/2}{\left(1 - e^{i\alpha}/2\right)\left(1 - e^{-i\alpha}/2\right)} \tag{6}$$

$$= \frac{1 - e^{-i\alpha}/2}{5/4 - \cos\alpha} \tag{7}$$

$$= \frac{1 - (\cos\alpha)/2}{5/4 - \cos\alpha} + i\frac{(\sin\alpha)/2}{5/4 - \cos\alpha} \tag{8}$$

We can treat this as a parametric form of a curve in the complex plane, with

$$\begin{aligned} x(\alpha) &= \frac{1 - (\cos\alpha)/2}{5/4 - \cos\alpha} \\ y(\alpha) &= \frac{(\sin\alpha)/2}{5/4 - \cos\alpha} \end{aligned} \tag{9}$$

with $0 \le \alpha < 2\pi$. The resulting curve is shown in Fig. 1.

This looks promising, as the circle does appear to be given by 1. To be sure, we need to grind through the algebra. I used Maple to ease the task somewhat. I'll work with the square of 1 to make things easier. We need to find

$$\left|d-\frac{4}{3}\right|^2 = \left(\frac{1-(\cos\alpha)/2}{5/4-\cos\alpha}-\frac{4}{3}\right)^2+\left(\frac{(\sin\alpha)/2}{5/4-\cos\alpha}\right)^2 \tag{10}$$

$$= \left(\frac{3(1-(\cos\alpha)/2)-4(5/4-\cos\alpha)}{3(5/4-\cos\alpha)}\right)^2+\left(\frac{(\sin\alpha)/2}{5/4-\cos\alpha}\right)^2 \tag{11}$$

$$= \left(\frac{8-10\cos\alpha}{-15+12\cos\alpha}\right)^2+\left(\frac{(\sin\alpha)/2}{5/4-\cos\alpha}\right)^2 \tag{12}$$

$$= \frac{4(-4+5\cos\alpha)^2}{9(-5+4\cos\alpha)^2}+\frac{4\sin^2\alpha}{(-5+4\cos\alpha)^2} \tag{13}$$

$$= \frac{4}{9(-5+4\cos\alpha)^2}\left(16-40\cos\alpha+25\cos^2\alpha+9\sin^2\alpha\right) \tag{14}$$

$$= \frac{4}{9(-5+4\cos\alpha)^2}\left(16-40\cos\alpha+9\left(\cos^2\alpha+\sin^2\alpha\right)+16\cos^2\alpha\right) \tag{15}$$

$$= \frac{4}{9(-5+4\cos\alpha)^2}\left(25-40\cos\alpha+16\cos^2\alpha\right) \tag{16}$$

$$= \frac{4}{9}\frac{(-5+4\cos\alpha)^2}{(-5+4\cos\alpha)^2} \tag{17}$$

$$= \frac{4}{9} \tag{18}$$

Thus we have

$$\left|d-\frac{4}{3}\right|^2 = \frac{4}{9} \tag{19}$$

which gives us 1 as required.