L'HOPITAL'S RULE

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Post date: 2 December 2024.

L'Hôpital's rule is a useful theorem that you may have encountered in first year calculus courses. It states that

Theorem 1. *If $f(z)$ and $g(z)$ are analytic at a point z_0 and $f(z_0) = g(z_0) = 0$, but $g'(z_0) \neq 0$, then*

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{1}$$

Proof. The proof relies on noticing a little trick. Since $f(z_0) = g(z_0) = 0$ we have

$$f(z) = f(z) - f(z_0) \tag{2}$$

$$g(z) = g(z) - g(z_0) \tag{3}$$

We can therefore write the ratio

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)} \tag{4}$$

$$= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\times\left[\frac{g(z)-g(z_0)}{z-z_0}\right]^{-1} \tag{5}$$

The two factors on the RHS are definitions of the derivatives, so we get

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{6}$$

□

Note that the limiting operation implies the usual condition for limits of complex functions, namely that the limit must exist for all paths leading to z_0 and be unique.