LAGRANGE MULTIPLIERS

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The method of Lagrange multipliers is used when we wish to maximize or minimize a function subject to a constraint. In its most general form, the method can be described as follows.

For a function $f(\{x_j\})$ of a set of variables $\{x_j\}$ subject to a number of constraints $c_k(\{x_j\}) = 0$, we form the compound function

$$G \equiv f + \sum_k \lambda_k c_k \tag{1}$$

To maximize or minimize f subject to the constraints c_k, we take the derivatives with respect to all the x_j and all the λ_k and set them equal to zero, and then solve the resulting set of simultaneous equations for the x_j and λ_k. The parameters λ_k are called the *Lagrange multipliers*.

Example 1. We want to find the rectangle with the largest area that can be inscribed within an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2}$$

Since the ellipse is centred at the origin, any rectangle inscribed within it will have its corners at $(\pm x, \pm y)$ and have an area of $A = 4xy$. The problem is then to maximize $4xy$ (or equivalently, just xy) subject to the constraint $c(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. The Lagrange multiplier function is then

$$G = xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{3}$$

The three derivatives are

$$\frac{\partial G}{\partial x} = y + \frac{2x\lambda}{a^2} = 0 \tag{4}$$

$$\frac{\partial G}{\partial y} = x + \frac{2y\lambda}{b^2} = 0 \tag{5}$$

$$\frac{\partial G}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{6}$$

(The derivatives with respect to the multipliers λ_k will always just give us the constraint equations $c_k(\{x_j\}) = 0$.)

We can use the first two equations to eliminate λ and we get

$$\lambda = \pm\frac{ab}{2} \tag{7}$$

$$y = \pm\frac{b}{a}x \tag{8}$$

Substituting the latter equation into the constraint, we get

$$x = \pm\frac{a}{\sqrt{2}} \tag{9}$$

$$y = \pm\frac{b}{\sqrt{2}} \tag{10}$$

The four combinations of these solutions give the four corners of the rectangle, so the resulting area is $A = 2ab$.

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