LAGUERRE POLYNOMIALS - NORMALIZATION

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Reference: Arfken, George B. & Weber, Hans J. (2005), Mathematical Methods for Physicists, 6th Edition, Academic Press - Sec 13.2.

The associated Laguerre ODE turns up in physics in several places, most notably in the solution of the Schrödinger equation for the hydrogen atom. The ODE is

$$x\frac{d^2L_n^k(x)}{dx^2}+(k+1-x)\frac{dL_n^k(x)}{dx}+nL_n^k(x)=0 \tag{1}$$

where $L_n^k(x)$ is an associated Laguerre polynomial and k and n are constants. This equation is not self-adjoint, but can be made so by multiplying by $e^{-x}x^k$:

$$e^{-x}x^{k+1}\frac{d^2L_n^k(x)}{dx^2}+(k+1-x)e^{-x}x^k\frac{dL_n^k(x)}{dx}+ne^{-x}x^kL_n^k(x)=0 \tag{2}$$

The self-adjoint condition is that the derivative of the function multiplying the second derivative term equals the function multiplying the first derivative term. In this case

$$\begin{aligned}\frac{d}{dx}\left(e^{-x}x^{k+1}\right) &= -e^{-x}x^{k+1}+(k+1)e^{-x}x^k &(3)\\ &= (k+1-x)e^{-x}x^k &(4)\end{aligned}$$

so the self-adjoint condition is satisfied.

We have seen that a self-adjoint operator D operating on a function that satisfies certain boundary conditions is a Hermitian operator, and as such we can write the ODE in the form

$$DL_n^k(x)+\lambda w(x)L_n^k(x)=0 \tag{5}$$

where $w(x)$ is the weighting function and λ is the eigenvalue. In this case

$$w(x) = e^{-x}x^k \quad (6)$$
$$\lambda = n \quad (7)$$

This means that the associated Laguerre polynomials for distinct values of n must be orthogonal. That is if $m \neq n$:

$$\int_0^\infty e^{-x}x^k L_n^k(x) L_m^k(x)\,dx = 0 \quad (8)$$

Note that this condition relies on choosing the limits of integration so that the boundary conditions are satisfied, and in this case these boundary conditions are (see post on self-adjoint operators):

$$L_n^k(x) p_0 L_m^k(x)' - L_m^k(x) p_0 L_n^k(x)'\Big|_0^\infty = 0 \quad (9)$$

Here the function $p_0(x)$ is the coefficient of the second derivative term, so is

$$p_0(x) = e^{-x}x^{k+1} \quad (10)$$

which is zero at both limits. Since the $L_n^k(x)$ are polynomials, the exponential dominates the behaviour at infinity, so the boundary conditions are satisfied.

This shows that the Laguerre polynomials are orthogonal, provided they are integrated with the appropriate weighting function $w(x)$. What we need to find is the normalization condition; that is, we need to find the integral of the square of a given polynomial:

$$\int_0^\infty e^{-x}x^k \left(L_n^k(x)\right)^2 dx \quad (11)$$

To do this, we can make use of the *generating function* for the Laguerre polynomials. Generating functions are somewhat magical, and deriving them is often trickier than using them. In the case of the Laguerre polynomials, the generating function is obtained by the use of Cauchy's theorem for integrals in the complex plane, which would take us too far afield for a single post. So I will ask the reader to accept the formula and leave the derivation for another post.

But first, what exactly *is* the generating function? It is a compact formula which, when expanded in a series, allows individual Laguerre polynomials to be picked off as the coefficients of individual powers in the series. That is, we define the generating function $g(x,z)$ so that

$$g(x,z)=\sum_{n=0}^{\infty}L_n^k(x)z^n \tag{12}$$

The variable z here is just a dummy variable which allows the series expansion to be derived. The catch, of course, is how to obtain $g(x,z)$ so it gives just the right series. It is here that I'm asking the reader to accept the function on faith at this stage. The generating function for associated Laguerre polynomials is:

$$\frac{e^{-xz/(1-z)}}{(1-z)^{k+1}}=\sum_{n=0}^{\infty}L_n^k(x)z^n \tag{13}$$

(As I said, it's not something you can pull out of a hat.)

Now that we have the generating function, we can use it to derive the normalization condition. We square the above equation, multiply it by the weighting function and then integrate the result:

$$\int_0^{\infty}e^{-x}x^k\frac{e^{-2xz/(1-z)}}{(1-z)^{2k+2}}dx = \int_0^{\infty}e^{-x}x^k\left(\sum_{n=0}^{\infty}L_n^k(x)z^n\right)^2 dx \tag{14}$$

$$= \sum_{n=0}^{\infty}z^{2n}\int_0^{\infty}e^{-x}x^k\left(L_n^k(x)\right)^2 dx \tag{15}$$

The second line is obtained by using the orthogonality condition 8, since this causes all the cross terms to vanish out of the sum once they are integrated.

The integral on the left is not that hard to do, since the integration variable x appears only in the exponential and the x^k term. It can be evaluated by integrating by parts k times, and noting that the integrated term vanishes at each stage due to the boundary conditions. We get (I cheated and used Maple):

$$\int_0^{\infty}e^{-x}x^k\frac{e^{-2xz/(1-z)}}{(1-z)^{2k+2}}dx = \frac{k!}{[z(1-z)]^{k+1}}\left(2+\frac{1-z}{z}\right)^{-(k+1)} \tag{16}$$

$$= \frac{k!}{(1-z^2)^{k+1}} \tag{17}$$

where the final answer is obtained by putting everything over a common denominator and using $(1-z)(1+z)=1-z^2$.

Now what we need to do is expand this result as a series in powers of z and match up corresponding terms with the desired integral in the series above.

The series corresponding to this term is a negative binomial expansion, so we can use the formula, valid for any (even complex) number r (again, if you haven't encountered binomial expansions, please accept this on faith):

$$(t+y)^r = \sum_{n=0}^{\infty} \binom{r}{n} t^{r-n} y^n \tag{18}$$

In this case, the binomial coefficient must be generalized (since r can be negative and non-integer):

$$\binom{r}{n} = \frac{r(r-1)\ldots(r-n+1)}{n!} \tag{19}$$

In our case, $r = -k-1$, $t = 1$ and $y = -z^2$ so we get

$$\frac{k!}{(1-z^2)^{k+1}} = k! \sum_{n=0}^{\infty} \binom{-k-1}{n} (-z^2)^n \tag{20}$$

$$= k! \sum_{n=0}^{\infty} \binom{-k-1}{n} (-1)^n z^{2n} \tag{21}$$

so comparing coefficients of z^{2n} on both sides, the normalization integral comes out to

$$\int_0^{\infty} e^{-x} x^k \left(L_n^k(x)\right)^2 dx = k! \binom{-k-1}{n} (-1)^n \tag{22}$$

Multiplying this out we get

$$\binom{-k-1}{n} = \frac{(-k-1)(-k-2)\ldots(-k-1-n+1)}{n!} \tag{23}$$

$$= (-1)^n \frac{(k+1)(k+2)\ldots(k+n)}{n!} \tag{24}$$

$$= (-1)^n \frac{(n+k)!}{n!k!} \tag{25}$$

Plugging this back into 22 we get

$$\int_0^{\infty} e^{-x} x^k \left(L_n^k(x)\right)^2 dx = (-1)^{2n} \frac{(n+k)!}{n!} \tag{26}$$

$$= \frac{(n+k)!}{n!} \tag{27}$$

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