LAURENT SERIES - EXAMPLES 1

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Here is an example of finding the Laurent series in three different domains. The function we're considering is

$$f(z)=\frac{z}{(z+1)(z-2)} \tag{1}$$

We first split it into partial fractions to get

$$f(z)=\frac{2}{3(z-2)}+\frac{1}{3(z+1)} \tag{2}$$

We first consider the domain $|z|<1$. In this case we can write both terms as geometric series. For the first term:

$$\frac{2}{3(z-2)}=\frac{2}{3}\times\frac{-1}{2(1-z/2)} \tag{3}$$

$$=-\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{k} \tag{4}$$

For the second term:

$$\frac{1}{3(z+1)}=\frac{1}{3}\sum_{k=0}^{\infty}(-z)^{k} \tag{5}$$

Thus the Laurent series is

$$f(z)=\frac{1}{3}\sum_{k=0}^{\infty}\left[(-1)^{k}-\frac{1}{2^{k}}\right]z^{k} \tag{6}$$

$$=-\frac{1}{2}z+\frac{1}{4}z^{2}-\frac{3}{8}z^{3}+\frac{5}{16}z^{4}-\frac{11}{32}z^{5}+\ldots \tag{7}$$

In this case there are no negative powers of z, so the series is also just a Taylor series.

Now we consider the domain $1<|z|<2$. In this case, to use the geometric series, we can use 4 directly since $|z/2|<1$, but we need to modify the series for the second term so we have a geometric series of a quantity with modulus less than 1. We have

$$\frac{1}{3(z+1)} = \frac{1}{3}\times\frac{1}{z}\frac{1}{(1+1/z)} \tag{8}$$

$$= \frac{1}{3z}\sum_{k=0}^{\infty}\left(\frac{-1}{z}\right)^k \tag{9}$$

$$= \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} \tag{10}$$

The Laurent series here is

$$f(z) = \frac{1}{3}\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{z^{k+1}} - \left(\frac{z}{2}\right)^k\right] \tag{11}$$

$$= -\frac{1}{3}+\frac{1}{3z}-\frac{z}{6}-\frac{1}{3z^2}-\frac{z^2}{12}+\frac{1}{3z^3}-\frac{z^3}{24}+\ldots \tag{12}$$

Here we have an infinite number of both positive and negative powers of z.

Finally, we consider $|z|>2$. We can now use 10 for the $1/[3(z+1)]$ term, but we need to provide another series for $2/[3(z-2)]$. We have

$$\frac{2}{3(z-2)} = \frac{2}{3z}\times\frac{1}{(1-2/z)} \tag{13}$$

$$= \frac{2}{3z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k \tag{14}$$

$$= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^{k+1} \tag{15}$$

The Laurent series here is

$$f(z) = \frac{1}{3}\sum_{k=0}^{\infty}\left[(-1)^k+2^{k+1}\right]\frac{1}{z^{k+1}} \tag{16}$$

$$= \frac{1}{z}+\frac{1}{z^2}+\frac{3}{z^3}+\frac{5}{z^4}+\frac{11}{z^5}+\frac{21}{z^6}+\frac{43}{z^7}+\ldots \tag{17}$$

Here all the powers of z are negative.

I checked the series in Maple by adding up the first 50 terms and comparing with 1.