LAURENT SERIES - EXAMPLES 2

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Here are a few more examples of calculating Laurent series.

Example 1. In the domain $0<|z-4|<4$ find the Laurent series for

$$f(z)=\frac{z+1}{z(z-4)^3} \tag{1}$$

The first step is to convert this to partial fractions. Referring back to the earlier post on partial fractions, we have

$$R_{1,4}(z)=\frac{A_0^{(1)}}{z}+\frac{A_0^{(2)}}{(z-4)^3}+\frac{A_1^{(2)}}{(z-4)^2}+\frac{A_2^{(2)}}{(z-4)} \tag{2}$$

To find the constants, we have

$$A_0^{(1)} = zR_{1,4}(z)\Big|_{z=0} \tag{3}$$

$$= -\frac{1}{64} \tag{4}$$

$$A_0^{(2)} = (z-4)^3 R_{1,4}(z)\Big|_{z=4} \tag{5}$$

$$= \frac{5}{4} \tag{6}$$

$$A_1^{(2)} = \frac{d}{dz}\left[(z-4)^3 R_{1,4}(z)\right]\Big|_{z=4} \tag{7}$$

$$= \left[\frac{1}{z}-\frac{z+1}{z^2}\right]\Big|_{z=4} \tag{8}$$

$$= -\frac{1}{16} \tag{9}$$

$$2A_2^{(2)} = \frac{d^2}{dz^2}\left[(z-4)^3 R_{1,4}(z)\right]\Big|_{z=4} \tag{10}$$

$$= -\frac{2}{z^2}+\frac{2(z+1)}{z^3}\Big|_{z=4} \tag{11}$$

$$= \frac{1}{32} \tag{12}$$

$$A_2^{(2)} = \frac{1}{64} \tag{13}$$

The partial fraction decomposition is therefore

$$f(z) = -\frac{1}{64z} + \frac{5}{4(z-4)^3} - \frac{1}{16(z-4)^2} + \frac{1}{64(z-4)} \tag{14}$$

The last three terms are already in the correct form for the Laurent series, so we need to convert the first term. We have

$$\frac{1}{z} = \frac{1}{4} \times \frac{1}{1+(z-4)/4} \tag{15}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{z-4}{4}\right)^k \tag{16}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k (z-4)^k}{4^{k+1}} \tag{17}$$

The Laurent series is thus

$$f(z) = \frac{5}{4(z-4)^3} - \frac{1}{16(z-4)^2} + \frac{1}{64(z-4)} - \frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^k (z-4)^k}{4^{k+1}} \tag{18}$$

$$= \frac{5}{4(z-4)^3} - \frac{1}{16(z-4)^2} + \frac{1}{64(z-4)} + \sum_{k=0}^{\infty}\frac{(-1)^{k+1} (z-4)^k}{4^{k+4}} \tag{19}$$

Example 2. In the domain $|z| > 0$, find the Laurent series for

$$f(z) = \frac{\sin(2z)}{z^3} \tag{20}$$

We start with the series for $\sin(2z)$:

$$\sin(2z) = 2z - \frac{4}{3}z^3 + \frac{4}{15}z^5 - \frac{8}{315}z^7 + \ldots \tag{21}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(2z)^{2k+1} \tag{22}$$

We can divide the series term by term by z^3 to get

$$\frac{\sin(2z)}{z^3} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} 2^{2k+1} z^{2k-2} \tag{23}$$

$$= \frac{2}{z^2} - \frac{4}{3} + \frac{4}{15} z^2 - \frac{8}{315} z^4 + \dots \tag{24}$$

Example 3. In the domain $|z| > 0$, find the Laurent series for

$$f(z) = z^2 \cos\left(\frac{1}{3z}\right) \tag{25}$$

The series for $\cos\left(\frac{1}{3z}\right)$ is

$$\cos\left(\frac{1}{3z}\right) = 1 - \frac{1}{(3z)^2 2!} + \frac{1}{(3z)^4 4!} - \frac{1}{(3z)^6 6!} + \dots \tag{26}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(3z)^{2k} (2k)!} \tag{27}$$

Multiplying term by term by z^2 gives

$$z^2 \cos\left(\frac{1}{3z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{2k} z^{2k-2} (2k)!} \tag{28}$$

$$= z^2 - \frac{1}{18} + \frac{1}{1944 z^2} - \frac{1}{524880 z^4} + \dots \tag{29}$$

The series does converge for all $|z| > 0$, although for very small values of $|z|$, we do need quite a few terms in the series.