LAURENT SERIES

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The Taylor series is used for finding a series expansion of an analytic function within its circle of convergence. However, if we want a series expansion of a function near a singularity, we need to use a generalized form of the Taylor series, known as a *Laurent series*. The relevant theorem is given in Section 5.5 of Saff and Snider, and may be stated as

Theorem 1. *Consider a function $f(z)$ that is analytic in the annulus $r<|z-z_0|<R$ for some fixed radii r and R. Then f can be expressed in this annulus as the series*

$$f(z)=\sum_{j=-\infty}^{\infty}a_j(z-z_0)^j \tag{1}$$

where the coefficients are given by the contour integral

$$a_j=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^{j+1}}d\zeta \tag{2}$$

where C is any positively oriented simple closed contour lying in the annulus and containing z_0 in its interior. The series 1 is called the Laurent series.

The proof of this theorem is quite involved, and is given in Saff and Snider, Section 5.5.

The important thing to note about the series 1 is that it contains negative powers of $(z-z_0)$ as well as the positive powers found in a Taylor series. In the case where $f(z)$ is analytic at $z=z_0$, the negative powers vanish and the series reduces to a Taylor series.

In the following examples, we find the Laurent series of

$$f(z)=\frac{1}{z+z^2} \tag{3}$$

in various domains.

Example 1. The domain is $0<|z|<1$. We start by writing f in partial fractions. We have

$$\frac{1}{z+z^2}=\frac{1}{z(1+z)}=\frac{1}{z}-\frac{1}{1+z} \tag{4}$$

In the domain $0<|z|<1$, we can expand the last term as a geometric series.

$$\frac{1}{1+z}=\frac{1}{1-(-z)}=\sum_{k=0}^{\infty}(-z)^k \tag{5}$$

The Laurent series is then

$$f(z)=\frac{1}{z}-\sum_{k=0}^{\infty}(-z)^k \tag{6}$$

$$=\frac{1}{z}-1+z-z^2+z^3-\ldots \tag{7}$$

Example 2. The domain is $|z|>1$. In this case, we can't use the geometric series in 7 since it doesn't converge for $|z|>1$. However, we can use a slight trick to rearrange things so we *can* use a geometric series.

$$\frac{1}{1+z}=\frac{1}{z}\times\frac{1}{1+1/z} \tag{8}$$

The $1/z$ term is now less than 1 in magnitude, so we can write

$$\frac{1}{1+1/z}=\frac{1}{1-(-1/z)} \tag{9}$$

$$=\sum_{k=0}^{\infty}\left(\frac{-1}{z}\right)^k \tag{10}$$

We can write 4 as

$$\frac{1}{z+z^2}=\frac{1}{z}-\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{-1}{z}\right)^k \tag{11}$$

$$=\frac{1}{z}-\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} \tag{12}$$

$$=\sum_{k=2}^{\infty}\left(\frac{-1}{z}\right)^k \tag{13}$$

$$=\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\ldots \tag{14}$$

Example 3. The domain is $0 < |z+1| < 1$. In this case, we want to get the partial fraction expression 4 in terms of $z+1$. We can modify the first term to get

$$\frac{1}{z} = \frac{1}{z+1-1} \tag{15}$$

$$= \frac{-1}{1-(z+1)} \tag{16}$$

$$= -\sum_{k=0}^{\infty}(z+1)^k \tag{17}$$

Therefore

$$f(z) = \frac{1}{z} - \frac{1}{1+z} \tag{18}$$

$$= -\sum_{k=0}^{\infty}(z+1)^k - \frac{1}{1+z} \tag{19}$$

$$= -\frac{1}{1+z} - 1 - (z+1) - (z+1)^2 - \dots \tag{20}$$

Example 4. The domain is $|z+1| > 1$. We can now write the $1/z$ term in the partial fraction expression as

$$\frac{1}{z} = \frac{1}{(z+1)} \times \frac{1}{1-1/(z+1)} \tag{21}$$

$$= \frac{1}{(z+1)}\sum_{k=0}^{\infty}\frac{1}{(z+1)^k} \tag{22}$$

$$= \sum_{k=1}^{\infty}\frac{1}{(z+1)^k} \tag{23}$$

We therefore get

$$f(z) = \frac{1}{z} - \frac{1}{1+z} \tag{24}$$

$$= \sum_{k=1}^{\infty}\frac{1}{(z+1)^k} - \frac{1}{(z+1)} \tag{25}$$

$$= \sum_{k=2}^{\infty}\frac{1}{(z+1)^k} \tag{26}$$

To check these results, I had Maple sum the first 50 or so terms of each series and compared the result with $1/\left(z+z^2\right)$. Fortunately, the results agreed with each other.

PINGBACKS

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