LEGENDRE EQUATION - LEGENDRE POLYNOMIALS

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A differential equation that occurs frequently in physics (as part of the solution of Laplace's equation, which occurs in such areas as electrodynamics and quantum mechanics, among others) is *Legendre's equation*. In this post, we'll have a look at the equation and some of the properties of its simplest solutions: the *Legendre polynomials*.

The equation occurs while solving Laplace's equation (which we'll consider in other posts) in spherical coordinates. Although the origins of the equation are important in the physical applications, for our purposes here we need concern ourselves only with the equation itself, which is usually first encountered in the following form:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{1}$$

where l and m are constants and the angle θ is the spherical coordinate which can range over the interval $[0,\pi]$ and serves as the independent variable in the equation. The problem is to determine the function $P(\theta)$.

The first step in solving the equation is usually to transform the variable to $x\equiv\cos\theta$. Under this transformation:

$$dx = -\sin\theta d\theta \tag{2}$$

$$\frac{dP}{d\theta} = -\sin\theta\frac{dP}{dx} \tag{3}$$

$$\sin^2\theta = 1-x^2 \tag{4}$$

so the equation becomes

$$\frac{1}{\sin\theta}(-\sin\theta)\frac{d}{dx}\left(-\sin^2\theta\frac{dP}{dx}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{5}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{6}$$

The solutions $P(x) = P(\cos\theta)$ of this equation are called *associated Legendre functions*. Before we dive in and try to find these general solutions, however, it is a bit easier if we look at the simpler case where $m = 0$:

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + l(l+1)P = 0 \tag{7}$$

Since the coefficients of P and its derivatives are all polynomials in x, a standard solution technique to try is that of proposing P as a power series (much as we did in solving the harmonic oscillator system in quantum mechanics). That is, we try a solution of the form

$$P(x) = \sum_{j=0}^{\infty} a_j x^j \tag{8}$$

where the a_j are coefficients to be determined. To substitute this into 7, we need to work out the various terms and derivatives.

$$\frac{dP}{dx} = \sum_{j=0}^{\infty} j a_j x^{j-1} \tag{9}$$

$$(1-x^2)\frac{dP}{dx} = \sum_{j=0}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} j a_j x^{j+1} \tag{10}$$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) = \sum_{j=0}^{\infty} j(j-1) a_j x^{j-2} - \sum_{j=0}^{\infty} j(j+1) a_j x^j \tag{11}$$

$$\begin{aligned}\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + l(l+1)P &= \sum_{j=0}^{\infty} j(j-1) a_j x^{j-2} - \sum_{j=0}^{\infty} j(j+1) a_j x^j + \sum_{j=0}^{\infty} l(l+1) a_j x^j \\ &= \sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} - j(j+1)a_j + l(l+1)a_j] x^j \\ &= 0\end{aligned} \tag{12}$$

where in the penultimate line we rearranged the first term so that both terms were summing over the same power of x.

If a series such as this is to be zero everywhere, then every term in the series must be zero, which provides a condition on the coefficients:

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+1)(j+2)} a_j \tag{13}$$

Since this recursion formula relates each coefficient to one *two* steps before it, we see we need to specify two of the coefficients to get the formula started. This is what we would expect, since a second order differential equation should have two constants that need specifying. So to start things off, we need to specify a_0 and a_1.

Now in order for this to be a sensible solution, the series needs to converge for all possible values of x. Remember that $x = \cos\theta$, so the range of possible values of x is $x \in [-1, 1]$. If the series is infinite, then for large j the recursion formula is approximately

$$a_{j+2} \approx a_j \tag{14}$$

so the series behaves like the geometric series $a\sum_{j=0}^{\infty} x^j$. Most mathematics textbooks deal with the geometric series and show that if the series is finite, it can be written in closed form as

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} \tag{15}$$

In the limit of $n \to \infty$, this formula converges to

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \tag{16}$$

provided $|x| < 1$. In our case, however, we need the series to converge for $x = \pm 1$ as well.

The way out of this dilemma is to require the series to terminate, so that $a_j = 0$ for all j greater than some finite value j_{max}. From the recursion formula, we see this can be arranged if $j_{max}(j_{max}+1) = l(l+1)$, so $j_{max} = l$ or $-l-1$. Since j_{max} must be a non-negative integer, only one of these solutions is acceptable, so we might as well concentrate on values of l that are non-negative integers (if $l < 0$, the constant $l(l+1)$ is still non-negative anyway, and that constant is all that appears in the original equation so we don't lose anything by restricting to $l \geq 0$). It then follows that if l is even, the highest value of j for which $a_j \neq 0$ must also be even, which in turn requires that all a_j for odd j must be zero. Conversely, if l is odd, then the even sub-series must be zero. That is, depending on the value of l, we must choose either $a_0 = 0$ or $a_1 = 0$.

Thus the solutions to 7 turn out to be polynomials in x, the forms of which are determined by the constant l, which must be a non-negative integer in order for the solutions to converge over the entire range of x. The first few such polynomials, called *Legendre polynomials*, are (taking $a_0 = 1$ or $a_1 = 1$ to get the series started; the subscript l on P_l is the value of l in each case)

$$P_0 = 1 \tag{17}$$

$$P_1 = x \tag{18}$$

$$P_2 = 1 - 3x^2 \tag{19}$$

$$P_3 = x - \frac{5}{3}x^3 \tag{20}$$

Since the differential equation is linear in P, each polynomial above can be multiplied by any constant and still be a solution, and the actual constant chosen depends on the normalization desired. This is equivalent to choosing a different value for a_0 or a_1 which, since these coefficients are arbitrary, is permissible. One convention is to normalize the polynomials so that $P_l(1) = 1$ for all l, and in that case, the polynomials start off as follows:

$$P_0 = 1 \tag{21}$$

$$P_1 = x \tag{22}$$

$$P_2 = \frac{1}{2}(3x^2 - 1) \tag{23}$$

$$P_3 = \frac{1}{2}(5x^3 - 3x) \tag{24}$$

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