LEGENDRE POLYNOMIALS - GEOMETRIC ORIGIN

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We've seen how Legendre polynomials arise from a simplified version of Legendre's differential equation. Here we have a look at how the polynomials can be generated from a simple geometric situation.

We start with an ordinary triangle with sides of lengths a, b and c, and define the angle θ between sides a and b. Then the cosine rule (a generalization of Pythagoras's theorem) says:

$$c=\sqrt{a^2+b^2-2ab\cos\theta} \tag{1}$$

If the triangle is right-angled, then $\theta=\pi/2$ and we get back Pythagoras's theorem.

Now if we consider the reciprocal of c (encountered in physics in any situation where some phenomenon depends on the inverse of the distance, as in the electrostatic or gravitational potential functions), we get

$$\begin{aligned}\frac{1}{c} &= \left(a^2+b^2-2ab\cos\theta\right)^{-1/2} &(2)\\ &= \frac{1}{b}\left(1+(a/b)^2-2(a/b)\cos\theta\right)^{-1/2} &(3)\end{aligned}$$

Now suppose we consider triangles where $b>a$. We can expand the term in brackets in a Taylor series in the quantity (a/b). To make the notation easier, we define $x\equiv\cos\theta$ and $t\equiv a/b$. Then the Taylor series will have the general form

$$\left(1+t^2-2xt\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n \tag{4}$$

This formula can be used as the starting point for a study of the Legendre polynomials if we *define* the quantities $P_n(x)$ to be the Legendre polynomials. Obviously, if we do this, we need to demonstrate that they are the same polynomials that turn up as the solutions to Legendre's differential equation, but we'll leave that to another post. What we'll do here is use this definition to derive an explicit formula for calculating $P_n(x)$.

First, we'll have a look at the Taylor series for the function $f(u)=(1-u)^{-1/2}$. Remember that the Taylor series about the reference point $u=0$ has the form

$$f(u)=\sum_{n=0}^{\infty}\frac{u^n}{n!}f^{(n)}(0) \tag{5}$$

where $f^{(n)}(0)$ is the n^{th} derivative of $f(u)$ evaluated at $u=0$. Calculating the first few derivatives of $f(u)=(1-u)^{-1/2}$ we get

$$f(u)=(1-u)^{-1/2} \tag{6}$$

$$f^{(1)}(u)=\frac{1}{2}(1-u)^{-3/2} \tag{7}$$

$$f^{(2)}(u)=\frac{3}{2}\frac{1}{2}(1-u)^{-5/2} \tag{8}$$

$$f^{(3)}(u)=\frac{5}{2}\frac{3}{2}\frac{1}{2}(1-u)^{-7/2} \tag{9}$$

It's fairly obvious that a pattern is forming, and the general formula for the n^{th} derivative is

$$f^{(n)}(u)=\frac{(2n-1)!!}{2^n}(1-u)^{-(2n+1)/2} \tag{10}$$

where the notation $(2n-1)!!$ is a *double factorial* which means the product of every second number from $2n-1$ down to 1. That is

$$(2n-1)!!=(2n-1)(2n-3)(2n-5)\ldots(3)(1) \tag{11}$$

Thus the derivatives evaluated at $u=0$ are

$$f^{(n)}(0)=\frac{(2n-1)!!}{2^n} \tag{12}$$

and the Taylor series is

$$f(u)=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}u^n \tag{13}$$

We've separated out the first term (1) in order to avoid a negative factorial from the $(2n-1)!!$ term.

We can get rid of the double factorial by noting that

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} \tag{14}$$
$$= \frac{(2n)!}{2^n n!} \tag{15}$$

The first line follows since the numerator is the product of all integers up to $2n$ and the denominator cancels off all the even integers, leaving the product of all the odd ones. The last line follows since

$$(2n)!! = (2n)(2n-2)(2n-4)\ldots(2) \tag{16}$$
$$= (2n)(2(n-1))(2(n-2))\ldots.(2(1)) \tag{17}$$
$$= 2^n n! \tag{18}$$

Thus we can write the Taylor series as

$$f(u) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} u^n \tag{19}$$

Using this in 4, we get

$$(1-(2xt-t^2))^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} (2xt-t^2)^n \tag{20}$$

Although it is possible to use this formula to pick out individual Legendre polynomials, it isn't very convenient, since we need to find all terms in a particular power of t to get the corresponding polynomial. However, the factor $(2xt-t^2)^n$ is an ordinary binomial, so we can use the binomial theorem to expand it.

The binomial theorem states

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{21}$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{22}$$

where $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{23}$$

The second line 22 in the theorem above points out that the expansion is symmetric in a and b.

With $a = 2xt$ and $b = -t^2$, we have

$$(2xt - t^2)^n = \sum_{k=0}^{n} \binom{n}{k} (2xt)^{n-k}(-t^2)^k \tag{24}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2x)^{n-k} t^{n+k} \tag{25}$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(-1)^k (2x)^{n-k} t^{n+k} \tag{26}$$

Substituting this back into 20 we get

$$(1-(2xt-t^2))^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(-1)^k (2x)^{n-k} t^{n+k} \tag{27}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(2n)!}{2^{2n} n! k!(n-k)!}(-1)^k (2x)^{n-k} t^{n+k} \tag{28}$$

We might still appear to be no further ahead, since we still need to pick out several separate terms to find all those terms where t has a particular exponent. However, there is a summation trick we can use to simplify things.

If we think of the term being summed as one entry in a matrix, we can write it using the notation

$$a_{nk} \equiv \frac{(2n)!}{2^{2n} n! k!(n-k)!}(-1)^k (2x)^{n-k} t^{n+k} \tag{29}$$

If the index n represents the row in the matrix and k the column, the summation in 28 extends over the lower triangular section of the matrix (that is, from the diagonal down to the lower left corner). The summation as written sums each row in turn from the first column out to the diagonal.

What we would like to do is to group the terms in the sum so that all terms with the same exponent for t are grouped together. That is, we are looking for those terms in the sum where $n + k = r$ for some particular value of r. Since we are considering the lower triangle of the matrix, these groups consist of

$$a_{00} \qquad r = 0 \tag{30}$$
$$a_{10} \qquad r = 1 \tag{31}$$
$$a_{20}, a_{11} \qquad r = 2 \tag{32}$$
$$a_{30}, a_{21} \qquad r = 3 \tag{33}$$
$$a_{40}, a_{31}, a_{22} \qquad r = 4 \tag{34}$$
$$\ldots \qquad r = \ldots \tag{35}$$

Thus for a given value of r, the matrix elements we want to select are of the form $a_{r-s,s}$ where $s = 0, 1, \ldots [r/2]$ where the notation $[r/2]$ means 'greatest integer less than or equal to $r/2$'.

We can therefore rearrange the sum by making the summation index substitutions $k = s$, $n = r - s$, and alter the summation ranges so that r runs from 0 to ∞ and s from 0 to $[r/2]$. We then get

$$(1-(2xt-t^2))^{-1/2} = \sum_{r=0}^{\infty} \sum_{s=0}^{[r/2]} \frac{(2r-2s)!}{2^{2r-2s}(r-s)!s!(r-2s)!}(-1)^s (2x)^{r-2s} t^r \tag{36}$$

We can now relabel the summation indexes back to k and n (since they are dummy indexes, we can call them whatever we like), and get

$$(1-(2xt-t^2))^{-1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{2^{2n-2k}(n-k)!k!(n-2k)!}(-1)^k (2x)^{n-2k} t^n \tag{37}$$

$$= \sum_{n=0}^{\infty} P_n(x) t^n \tag{38}$$

where the last line is by comparison with 4.

By the uniqueness of power series, the coefficients of each power of t must be equal on either side of the equation, so we get our explicit formula for the Legendre polynomials:

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{2^{2n-2k}(n-k)!k!(n-2k)!}(-1)^k (2x)^{n-2k} \tag{39}$$

$$= \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{2^{n}(n-k)!k!(n-2k)!}(-1)^k x^{n-2k} \tag{40}$$

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