LIMITS OF COMPLEX FUNCTIONS

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The limit of a complex function $f(z)$ as $z \to z_0$ is a bit more involved than the limit of a real function, since a complex function is defined over the complex plane, rather than the real axis, as for a real function $f(x)$. As a result, the approach of z to z_0 can occur from all different directions, or in fact by some convoluted path such as a spiral. To formalize the definition of the limit of a complex function, we state as follows.

Let $f(z)$ be a function defined in some neighbourhood of a point z_0, with the possible exception of z_0 itself. Then the limit of $f(z)$ as z approaches z_0 (by any path) is the complex number w_0, written as

$$\lim_{z \to z_0} f(z) = w_0 \tag{1}$$

or sometimes just as

$$f(z) \to w_0 \text{ as } z \to z_0 \tag{2}$$

In order for this limit to exist we require that

$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \tag{3}$$

In words, this states that if z is within some circle centred at z_0, then, provided this circle is small enough, $f(z)$ lies within a distance ε of the limit point w_0.

Example 1. Using this definition, we prove that $\lim_{z \to 1+i} (6z - 4) = 2 + 6i$.

The key to using the definition is to express $f(z) - w_0$ in terms of $z - z_0$. Here, this means we need to write $f(z) - w_0 = 6z - 4 - (2 + 6i)$ in terms of $z - z_0 = z - (1 + i)$. We have

$$6z - 4 - (2 + 6i) = 6z - 6 - 6i \tag{4}$$
$$= 6(z - (1 + i)) \tag{5}$$

Taking moduli, we have

$$|f(z)-w_0| = |6(z-(1+i))| \tag{6}$$
$$= 6|z-(1+i)| \tag{7}$$

Thus if we choose $\delta = \frac{\varepsilon}{6}$ so that $|z-(1+i)| < \frac{\varepsilon}{6}$, then

$$|f(z)-w_0| < 6\frac{\varepsilon}{6} = \varepsilon \tag{8}$$

Example 2. Show that $\lim_{z\to -i}\frac{1}{z} = i$.

This is a bit trickier that the previous example. We need to express $f(z)-w_0$ in terms of $z-z_0$, the latter of which in this case is $z-(-i) = z+i$. We have

$$f(z)-w_0 = \frac{1}{z} - i \tag{9}$$
$$= \frac{1}{z+i-i} - i\frac{z+i-i}{z+i-i} \tag{10}$$
$$= -i\frac{z+i}{z+i-i} \tag{11}$$

Taking moduli

$$|f(z)-w_0| = \frac{|z+i|}{|z+i-i|} \tag{12}$$

We need to find some δ so that if $|z+i| < \delta$, then $\frac{|z+i|}{|z+i-i|} < \varepsilon$. We are looking for z values that are close to $-i$, which means that if $|z+i| < \delta$, then z is within a small circle centred at $z_0 = (0,-i)$. This means that the point $z+i-i$ is at least a distance $1-\delta$ from z_0, so that the minimum value for $|z+i-i|$ is $1-\delta$. In other words,

$$|z+i-i| \geq 1-\delta \tag{13}$$

Therefore we can say that

$$\frac{|z+i|}{|z+i-i|} \leq \frac{\delta}{1-\delta} = \varepsilon \tag{14}$$

This will be true if

$$\delta = \frac{\varepsilon}{1+\varepsilon} \tag{15}$$

I know this looks like a lot of work for something which can be calculated by just taking $\frac{1}{-i}$ and rationalizing the denominator, but this is the way a formal limit is defined, so it's worth going through the motions for a couple of examples.

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