LINEAR MAPS, LINEAR OPERATORS AND COMMUTATORS

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1. Linear maps

Having looked at some of the properties of a vector space, we can now look at *linear maps*. A linear map T is defined as a function that maps one vector space V into another (possibly the same) vector space W, written as

$$T : V \to W \tag{1}$$

The linear map T must satisfy the two properties

(1) **Additivity**: $T(u+v) = Tu + Tv$ for all $u, v \in V$.
(2) **Homogeneity**: $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$. As usual, the field $\mathbb{F}$ is either the set of real $\mathbb{R}$ or complex numbers $\mathbb{C}$.

This definition of a linear map is general in the sense that the two vector spaces V and W can be any two vector spaces. In physics, it's more common to have $V = W$, and in such a case, the linear map T is called a *linear operator*.

2. Linear operators

With a couple of extra definitions, the set $\mathcal{L}(V)$ of all linear operators on V is itself a vector space, with the operators being the vectors. In order for this to be true, we need the following:

(1) **Zero operator**: A zero operator, written as just 0 (the same symbol now being used for three distinct objects: the scalar 0, the vector 0 and the operator 0; again the correct meaning is usually easy to deduce from the context) which has the property that the result of acting with 0 on any vector produces the 0 vector. That is $0v = 0$, where the 0 on the LHS is the zero operator and the 0 on the RHS is the zero vector.
(2) **Identity operator**: An identity operator I (sometimes written as 1) leaves any vector unchanged, so that $Iv = v$ for all $v \in V$.

With these definitions, $\mathcal{L}(V)$ is now a vector space, since it satisfies the distributive (additivity) and scalar multiplication (homogeneity) properties, contains an additive identity (the zero operator) and a multiplicative identity (the identity operator).

In addition, there is a natural definition of the multiplication of two linear operators S and T, written as ST. When a product operates on a vector $v \in V$, we just operate from right to left (not left to right!) in succession, so that

$$(ST)v = S(Tv) \tag{2}$$

The product of two operators produces another operator also in $\mathcal{L}(V)$, since this product also satisfies additivity and homogeneity:

$$\begin{aligned}
(ST)(u+v) &= S(T(u+v)) & (3)\\
&= S(Tu+Tv) & (4)\\
&= STu+STv & (5)\\
(ST)(\lambda v) &= S(T(\lambda v)) & (6)\\
&= S(\lambda Tv) & (7)\\
&= \lambda S(Tv) & (8)\\
&= \lambda(ST)v & (9)
\end{aligned}$$

A very important property of operator multiplication is that it is not necessarily commutative. In quantum mechanics, for example, the position and momentum operators do not commute. The non-commutativity is a fundamental mathematical property however, and can be seen in other examples that have nothing to do with quantum theory.

For example, consider the left shift operator L and right shift operator R, defined to act on the vector space consisting of infinite sequences of numbers. That is, our vector space V is such that

$$v = (x_1, x_2, x_3, \ldots) \tag{10}$$

where $x_i \in \mathbb{F}$. The shift operators have the following effects:

$$\begin{aligned}
Lv &= (x_2, x_3, \ldots) & (11)\\
Rv &= (0, x_1, x_2, x_3, \ldots) & (12)
\end{aligned}$$

The L operator removes the first element in the sequence, while the R operator inserts a 0 (number!) as the new first element in the sequence. Note that 0 is the only number we could insert into a sequence in order that R be a linear operator, since from additivity above, we must have $R0 = 0$. That is, if we start with $v = 0$ (the vector all of whose elements $x_i = 0$), then $R0$ must also give the zero vector.

The two products LR and RL produce different results:

$$LRv = L(0,x_1,x_2,x_3,\ldots) = (x_1,x_2,x_3,\ldots) = v \tag{13}$$
$$RLv = R(x_2,x_3,\ldots) = (0,x_2,x_3,\ldots) \neq v \tag{14}$$

3. COMMUTATORS

The difference $[L,R] \equiv LR - RL$ is called the *commutator* of the two operators L and R. For example, if we introduce the operator which projects out the first element in the sequence:

$$P_1 v \equiv P_1(x_1,x_2,x_3,\ldots) = (x_1,0,0,\ldots) \tag{15}$$
$$Iv - P_1 v = (x_1,x_2,x_3,\ldots) - (x_1,0,0,\ldots) \tag{16}$$
$$= (0,x_2,x_3,\ldots) = RLv \tag{17}$$

then we have

$$[L,R]v = Iv - (Iv - P_1 v) \tag{18}$$
$$= P_1 v \tag{19}$$

so that the commutator $[L,R]$ projects out the first element of a vector.

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