

LINEAR OPERATORS - CHANGE OF BASIS, TRACE AND DETERMINANT

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1. CHANGE OF BASIS

We've seen that the matrix representation of a linear operator depends on the basis we've chosen within a vector space V . With a linear operator T , we can operate on a vector v_j from one set $\{v\}$ of basis vectors. This operation will, in general, produce some other vector v'_j . Since any vector in V can be expressed as a linear combination of the basis vectors, we have

$$Tv_j = v'_j = \sum_{i=1}^n T_{ij}(\{v\})v_i \quad (1)$$

This equation defines the *matrix elements* $T_{ij}(\{v\})$ of the operator T in the basis $\{v\}$. If we used a different basis $\{u\}$, then we would could express v'_j in this new basis with the equation:

$$v'_j = \sum_{i=1}^n T_{ij}(\{u\})u_i \quad (2)$$

Since the vectors in $\{u\}$ are different from the vectors in $\{v\}$, we would expect that the coefficients required to write v'_j in terms of the two bases would be different. That is, we would expect that, in general

$$T_{ij}(\{u\}) \neq T_{ij}(\{v\}) \quad (3)$$

In fact, the two sets of matrix elements *are* the same if T converts one set of basis vectors $\{u\}$ into another $\{v\}$, as we'll see below.

We now look at how the matrix representation changes if we change the basis. In what follows, we'll consider two sets of basis vectors $\{v\}$ and $\{u\}$ and two operators A and B . Operator A transforms the basis $\{v\}$ into the basis $\{u\}$, while B does the reverse. That is

$$\begin{aligned} Av_i &= u_i \\ Bu_i &= v_i \end{aligned} \quad (4)$$

for all $i = 1, \dots, n$. From this definition, we can see that $A = B^{-1}$ and $B = A^{-1}$, since

$$u_i = Av_i = ABu_i \quad (5)$$

$$v_i = Bu_i = BAv_i \quad (6)$$

Theorem 1. *An operator (like A or B above) that transforms one set of basis vectors into another has the same matrix representation in both bases.*

Proof. In matrix form, we have (remember we're using the summation convention on repeated indices):

$$Av_i = A_{ji}(\{v\})v_j \quad (7)$$

$$Au_i = A_{ji}(\{u\})u_j \quad (8)$$

Note that the matrix elements depend on *different* bases in the two equations.

We can now operate with A again, using 4, to get

$$Au_i = A(Av_i) \quad (9)$$

$$= A(A_{ji}(\{v\})v_j) \quad (10)$$

$$= A_{ji}(\{v\})Av_j \quad (11)$$

$$= A_{ji}(\{v\})u_j \quad (12)$$

Comparing the last line with 8, we see that

$$A_{ji}(\{v\}) = A_{ji}(\{u\})$$

Since the matrix elements are just numbers, this means that the elements in the two matrices $A_{ji}(\{v\})$ and $A_{ji}(\{u\})$ are the same.

We could do the same analysis using the B operator with the same result:

$$B_{ji}(\{v\}) = B_{ji}(\{u\}) \quad (13)$$

□

We can now turn to the matrix representations of a general operator T in two different bases. In this case, T can perform any linear transformation, so it doesn't necessarily transform one set of basis vectors into another set of basis vectors. Consider first the case where T operates on each set of basis vectors given above:

$$Tv_i = T_{ji}(\{v\})v_j \quad (14)$$

$$Tu_i = T_{ji}(\{u\})u_j \quad (15)$$

Unless T is an operator like A or B above, in general $T_{ji}(\{v\}) \neq T_{ji}(\{u\})$. We can see how these two matrices are related by using operators A and B above to write

$$Tu_i = T(A_{ji}v_j) \quad (16)$$

$$= A_{ji}Tv_j \quad (17)$$

$$= A_{ji}T_{kj}(\{v\})v_k \quad (18)$$

$$= A_{ji}T_{kj}(\{v\})Bu_k \quad (19)$$

$$= A_{ji}T_{kj}(\{v\})A^{-1}u_k \quad (20)$$

$$= A_{ji}T_{kj}(\{v\})A_{pk}^{-1}u_p \quad (21)$$

$$= [A_{pk}^{-1}T_{kj}(\{v\})A_{ji}]u_p \quad (22)$$

$$= T_{pi}(\{u\})u_p \quad (23)$$

We don't need to specify the basis for the A or B matrices since the matrices are the same in both bases as we just saw above. The last line is just the expansion of Tu_i in terms of the $\{u\}$ basis. In the penultimate line, we see that the quantity in square brackets is the product of 3 matrices:

$$A_{pk}^{-1}T_{kj}(\{v\})A_{ji} = [A^{-1}T(\{v\})A]_{pi} \quad (24)$$

The required transformation is therefore

$$T(\{u\}) = A^{-1}T(\{v\})A \quad (25)$$

where $u_i = Av_i$.

As a check, note that if $T = A$ or $T = B = A^{-1}$, we reclaim the result in the theorem above, namely that $A(\{u\}) = A(\{v\})$ and $B(\{u\}) = B(\{v\})$.

2. TRACE AND DETERMINANT

The trace of a matrix is the sum of its diagonal elements, written as $\text{tr } T$. A useful property of the trace is that

$$\text{tr}(AB) = \text{tr}(BA) \quad (26)$$

We can prove this by looking at the components. If $C = AB$ then

$$C_{ij} = A_{ik}B_{kj} \quad (27)$$

The trace of C is the sum of its diagonal elements, written as C_{ii} , so

$$\operatorname{tr} C = \operatorname{tr} (AB) \quad (28)$$

$$= A_{ik} B_{ki} \quad (29)$$

$$= B_{ki} A_{ik} \quad (30)$$

$$= [BA]_{kk} \quad (31)$$

$$= \operatorname{tr} (BA) \quad (32)$$

From this we can generalize to the case of the trace of a product of any number of matrices and obtain the cyclic rule:

$$\operatorname{tr}(A_1 A_2 \dots A_n) = \operatorname{tr}(A_n A_1 A_2 \dots A_{n-1}) \quad (33)$$

Going back to 25, we have

$$\operatorname{tr} T(\{u\}) = \operatorname{tr}(A^{-1} T(\{v\}) A) \quad (34)$$

$$= \operatorname{tr}(A A^{-1} T(\{v\})) \quad (35)$$

$$= \operatorname{tr} T(\{v\}) \quad (36)$$

Thus the trace of any linear operator is invariant under a change of basis.

For the determinant, we have the results that the determinant of a product of matrices is equal to the product of the determinants, and the determinant of a matrix inverse is the reciprocal of the determinant of the original matrix (this is covered in most introductory linear algebra courses, but see Theorem 7.15 in Liesen & Mehrmann for a proof). Therefore

$$\det(T(\{u\})) = \det(A^{-1} T(\{v\}) A) \quad (37)$$

$$= \frac{\det A}{\det A} \det T(\{v\}) \quad (38)$$

$$= \det T(\{v\}) \quad (39)$$

Thus the determinant is also invariant under a change of basis.

REFERENCES

- (1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 10.
- (2) Liesen, Jörg & Mehrmann, Volker (2015), *Linear Algebra*, Chapter 7.
- (3) Landi, Giovanni & Zampini, Alessandro (2018), *Linear Algebra and Analytic Geometry for Physical Sciences*, Chapter 10.

- (4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

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