LINEAR OPERATORS - CHANGE OF BASIS, TRACE AND DETERMINANT

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1. Change of basis

We've seen that the matrix representation of a linear operator depends on the basis we've chosen within a vector space V. With a linear operator T, we can operate on a vector v_j from one set $\{v\}$ of basis vectors. This operation will, in general, produce some other vector v_j'. Since any vector in V can be expressed as a linear combination of the basis vectors, we have

$$Tv_j = v_j' = \sum_{i=1}^{n} T_{ij}(\{v\})\, v_i \tag{1}$$

This equation defines the *matrix elements* $T_{ij}(\{v\})$ of the operator T in the basis $\{v\}$. If we used a different basis $\{u\}$, then we would could express v_j' in this new basis with the equation:

$$v_j' = \sum_{i=1}^{n} T_{ij}(\{u\})\, u_i \tag{2}$$

Since the vectors in $\{u\}$ are different from the vectors in $\{v\}$, we would expect that the coefficients required to write v_j' in terms of the two bases would be different. That is, we would expect that, in general

$$T_{ij}(\{u\}) \neq T_{ij}(\{v\}) \tag{3}$$

In fact, the two sets of matrix elements *are* the same if T converts one set of basis vectors $\{u\}$ into another $\{v\}$, as we'll see below.

We now look at how the matrix representation changes if we change the basis. In what follows, we'll consider two sets of basis vectors $\{v\}$ and $\{u\}$ and two operators A and B. Operator A transforms the basis $\{v\}$ into the basis $\{u\}$, while B does the reverse. That is

$$\begin{aligned} Av_i &= u_i \\ Bu_i &= v_i \end{aligned} \tag{4}$$

for all $i = 1, \ldots, n$. From this definition, we can see that $A = B^{-1}$ and $B = A^{-1}$, since

$$u_i = Av_i = ABu_i \tag{5}$$
$$v_i = Bu_i = BAv_i \tag{6}$$

Theorem 1. *An operator (like A or B above) that transforms one set of basis vectors into another has the same matrix representation in both bases.*

Proof. In matrix form, we have (remember we're using the summation convention on repeated indices):

$$Av_i = A_{ji}(\{v\})\,v_j \tag{7}$$
$$Au_i = A_{ji}(\{u\})\,u_j \tag{8}$$

Note that the matrix elements depend on *different* bases in the two equations.

We can now operate with A again, using 4, to get

$$\begin{aligned} Au_i &= A(Av_i) && (9)\\ &= A(A_{ji}(\{v\})\,v_j) && (10)\\ &= A_{ji}(\{v\})\,Av_j && (11)\\ &= A_{ji}(\{v\})\,u_j && (12) \end{aligned}$$

Comparing the last line with 8, we see that

$$A_{ji}(\{v\}) = A_{ji}(\{u\})$$

Since the matrix elements are just numbers, this means that the elements in the two matrices $A_{ji}(\{v\})$ and $A_{ji}(\{u\})$ are the same.

We could do the same analysis using the B operator with the same result:

$$B_{ji}(\{v\}) = B_{ji}(\{u\}) \tag{13}$$

□

We can now turn to the matrix representations of a general operator T in two different bases. In this case, T can perform any linear transformation, so it doesn't necessarily transform one set of basis vectors into another set of basis vectors. Consider first the case where T operates on each set of basis vectors given above:

$$Tv_i = T_{ji}(\{v\})v_j \tag{14}$$
$$Tu_i = T_{ji}(\{u\})u_j \tag{15}$$

Unless T is an operator like A or B above, in general $T_{ji}(\{v\}) \neq T_{ji}(\{u\})$. We can see how these two matrices are related by using operators A and B above to write

$$Tu_i = T(A_{ji}v_j) \tag{16}$$
$$= A_{ji}Tv_j \tag{17}$$
$$= A_{ji}T_{kj}(\{v\})v_k \tag{18}$$
$$= A_{ji}T_{kj}(\{v\})Bu_k \tag{19}$$
$$= A_{ji}T_{kj}(\{v\})A^{-1}u_k \tag{20}$$
$$= A_{ji}T_{kj}(\{v\})A^{-1}_{pk}u_p \tag{21}$$
$$= \left[A^{-1}_{pk}T_{kj}(\{v\})A_{ji}\right]u_p \tag{22}$$
$$= T_{pi}(\{u\})u_p \tag{23}$$

We don't need to specify the basis for the A or B matrices since the matrices are the same in both bases as we just saw above. The last line is just the expansion of Tu_i in terms of the $\{u\}$ basis. In the penultimate line, we see that the quantity in square brackets is the product of 3 matrices:

$$A^{-1}_{pk}T_{kj}(\{v\})A_{ji} = \left[A^{-1}T(\{v\})A\right]_{pi} \tag{24}$$

The required transformation is therefore

$$T(\{u\}) = A^{-1}T(\{v\})A \tag{25}$$

where $u_i = Av_i$.

As a check, note that if $T = A$ or $T = B = A^{-1}$, we reclaim the result in the theorem above, namely that $A(\{u\}) = A(\{v\})$ and $B(\{u\}) = B(\{v\})$.

2. Trace and determinant

The trace of a matrix is the sum of its diagonal elements, written as tr T. A useful property of the trace is that

$$\text{tr }(AB) = \text{tr }(BA) \tag{26}$$

We can prove this by looking at the components. If $C = AB$ then

$$C_{ij} = A_{ik}B_{kj} \tag{27}$$

The trace of C is the sum of its diagonal elements, written as C_{ii}, so

$$\begin{aligned}\text{tr } C &= \text{tr } (AB) &(28)\\ &= A_{ik}B_{ki} &(29)\\ &= B_{ki}A_{ik} &(30)\\ &= [BA]_{kk} &(31)\\ &= \text{tr } (BA) &(32)\end{aligned}$$

From this we can generalize to the case of the trace of a product of any number of matrices and obtain the cyclic rule:

$$\text{tr}\left(A_1A_2\ldots A_n\right) = \text{tr}\left(A_nA_1A_2\ldots A_{n-1}\right) \tag{33}$$

Going back to 25, we have

$$\begin{aligned}\text{tr } T(\{u\}) &= \text{tr}\left(A^{-1}T(\{v\})A\right) &(34)\\ &= \text{tr}\left(AA^{-1}T(\{v\})\right) &(35)\\ &= \text{tr } T(\{v\}) &(36)\end{aligned}$$

Thus the trace of any linear operator is invariant under a change of basis.

For the determinant, we have the results that the determinant of a product of matrices is equal to the product of the determinants, and the determinant of a matrix inverse is the reciprocal of the determinant of the original matrix (this is covered in most introductory linear algebra courses, but see Theorem 7.15 in Liesen & Mehrmann for a proof). Therefore

$$\begin{aligned}\det\left(T(\{u\})\right) &= \det\left(A^{-1}T(\{v\})A\right) &(37)\\ &= \frac{\det A}{\det A}\det T(\{v\}) &(38)\\ &= \det T(\{v\}) &(39)\end{aligned}$$

Thus the determinant is also invariant under a change of basis.

References

(1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 10.
(2) Liesen, Jörg & Mehrmann, Volker (2015), *Linear Algebra,* Chapter 7.
(3) Landi, Giovanni & Zampini, Alessandro (2018), *Linear Algebra and Analytic Geometry for Physical Sciences*, Chapter 10.

(4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

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