LINEAR TRANSFORMATIONS AS MAPS

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Post date: 11 May 2025.

In mapping theory, a *linear transformation* from z space to w space is defined by

$$w = az + b \tag{1}$$

where a and b are complex constants. Such a transformation can be of three types, with a general transformation a combination of these three types.

(1) A *translation* has the form

$$w = z + b \tag{2}$$

That is, $a = 1$. This has the effect of shifting all points by the distance b (remember that b can be complex, so in the general case this is a shift in the complex plane).

(2) A *magnification* has the form

$$w = \rho z \tag{3}$$

where ρ is a positive, real constant. This has the effect of scaling the original domain up by the factor ρ (or down, if $\rho < 1$).

(3) A rotation has the form

$$w = e^{i\phi} z \tag{4}$$

where ϕ is a real constant. This rotates every point in the original domain by an angle ϕ about the origin.

Note that the term 'linear transformation' is bad terminology, as in other areas of mathematics, such a transformation obeys the rule $f(z_1 + z_2) = f(z_1) + f(z_2)$. This isn't true for the general transformation 1, since

$$f(z_1 + z_2) = a(z_1 + z_2) + b \tag{5}$$

$$\neq f(z_1) + f(z_2) \tag{6}$$

$$= (az_1 + b) + (az_2 + b) \tag{7}$$

unless $b = 0$.

Example 1. Find a linear transformation that maps the circle $|z|=1$ onto the circle $|w-5|=3$, such that the point $z=i$ on the original circle is mapped into $w=2$ on the transformed circle.

Our first guess might be to magnify the circle by a factor of $\rho=3$ and then translate it by $b=5$. However, if we do this we find

$$w_1 = 3z \tag{8}$$

$$w_2 = w_1+5=3z+5 \tag{9}$$

Plugging in $z=i$, however, transforms the point to $w=3i+5$. We need to rotate the circle before the translation, so we try

$$w_1 = 3z \tag{10}$$

$$w_2 = e^{i\phi}w_1 \tag{11}$$

$$w_3 = w_2+5 \tag{12}$$

$$= 3e^{i\phi}z+5 \tag{13}$$

We now need to find ϕ, which we do by requiring $w=2$ when $z=i$.

$$3e^{i\phi}i+5 = 2 \tag{14}$$

$$3e^{i(\phi+\pi/2)} = -3 \tag{15}$$

Therefore

$$e^{i(\phi+\pi/2)} = -1 \tag{16}$$

so

$$\phi = \frac{\pi}{2} \tag{17}$$

The final transformation is

$$w = 3e^{i\pi/2}z+5 = 3iz+5 \tag{18}$$

As a check, we see that

$$|w-5| = |3iz+5-5| \tag{19}$$

$$= |3iz| \tag{20}$$

$$= 3|z| \tag{21}$$

$$= 3 \tag{22}$$

since $|z|=1$ on the original circle.

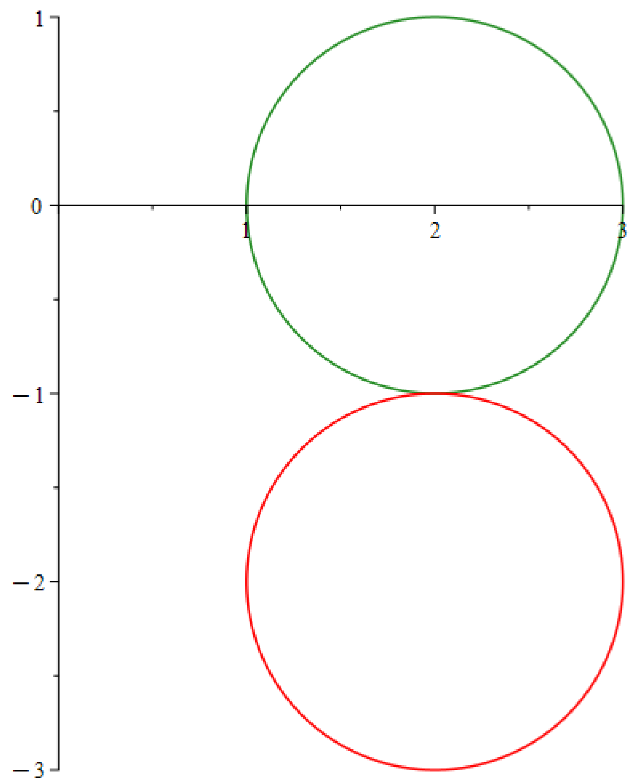

FIGURE 1. Original circle $|z-2|=1$ in green (top). Transformed circle $|w+2i-2|=\frac{1}{3}$ in red (bottom).

Example 2. Find the image of the circle $|z-2|=1$ under the transformation $w=z-2i$. This is a simple translation, which we can write as

$$z=w+2i \tag{23}$$

so the circle transforms to

$$|w+2i-2|=1 \tag{24}$$

This is a circle with centre at $w_0=2-2i$ and radius 1. The interior of the original circle becomes the interior of the transformed circle. A plot of the two circles is in Fig. 1.

Example 3. Find the image of the circle $|z-2|=1$ under the transformation $w=3iz$. This is a combination of a magnification and rotation. We have

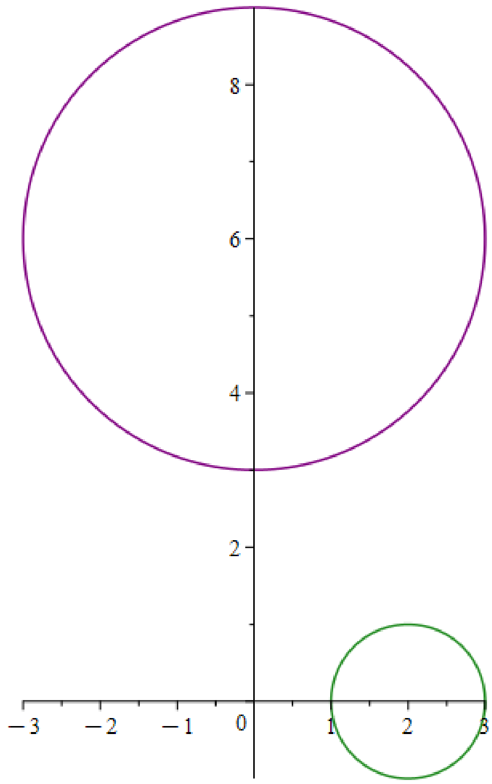

FIGURE 2. Original circle $|z-2|=1$ in green (bottom). Transformed circle $|w-6i|=3$ in purple (top).

$$z=\frac{w}{3i}=\frac{1}{3}e^{-i\pi/2}w \tag{25}$$

Therefore

$$\left|\frac{1}{3}e^{-i\pi/2}w-2\right|=1 \tag{26}$$

$$\left|e^{-i\pi/2}w-6\right|=3 \tag{27}$$

$$\left|e^{-i\pi/2}\left(w-6e^{i\pi/2}\right)\right|=3 \tag{28}$$

$$|w-6i|=3 \tag{29}$$

The transformed circle has its centre at $w_0=6i$ and a radius of 3. The interior points of the original circle are mapped to the interior of the transformed circle, but all distances are multiplied by 3. A plot of the two circles is in Fig. 2.

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