LINEARLY INDEPENDENT EXPONENTIALS

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Theorem 1. *Given a set of m distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ where $\lambda_k \neq \lambda_\ell$ for $k \neq \ell$ the functions $e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_m z}$ are linearly independent on the complex plane. That is if*

$$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e^{\lambda_m z} = 0 \tag{1}$$

the only solution is

$$c_1 = c_2 = \ldots = c_m = 0 \tag{2}$$

Proof. We can prove this by induction. First, we note that if $m = 1$, then

$$c_1 e^{\lambda_1 z} = 0 \tag{3}$$

has the unique solution $c_1 = 0$, since the exponential function is never zero.

Next we assume that 2 is the only solution of 1 for a given value of m, and use this to prove that adding another term $c_{m+1} e^{\lambda_{m+1} z}$ to the LHS of 1 requires that $c_{m+1} = 0$. That is, we start with

$$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e^{\lambda_m z} + c_{m+1} e^{\lambda_{m+1} z} = 0 \tag{4}$$

We divide through by $e^{\lambda_1 z}$ to get

$$c_1 + c_2 e^{(\lambda_2 - \lambda_1) z} + \ldots + c_m e^{(\lambda_m - \lambda_1) z} + c_{m+1} e^{(\lambda_{m+1} - \lambda_1) z} = 0 \tag{5}$$

Next, we take the derivative with respect to z to get

$$c_2 (\lambda_2 - \lambda_1) e^{(\lambda_2 - \lambda_1) z} + \ldots + c_m (\lambda_m - \lambda_1) e^{(\lambda_m - \lambda_1) z} + c_{m+1} (\lambda_{m+1} - \lambda_1) e^{(\lambda_{m+1} - \lambda_1) z} = 0 \tag{6}$$

The inductive assumption is that 2 is true for the value m so 6 reduces to

$$c_{m+1} (\lambda_{m+1} - \lambda_1) e^{(\lambda_{m+1} - \lambda_1) z} = 0 \tag{7}$$

Also, one of the assumptions in the theorem is that all the λ_ks are different, so $\lambda_{m+1} - \lambda_1 \neq 0$, therefore we must have

$$c_{m+1} = 0 \tag{8}$$

Therefore 2 is true for $m+1$, thus the functions $e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_m z}, e^{\lambda_{m+1} z}$ are linearly independent. $\square$