LIOUVILLE'S THEOREM

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One consequence of the Cauchy estimates is *Liouville's theorem*:

Theorem 1. *The only bounded entire functions are the constant functions.*

The proof is given in Saff and Snider's book, Section 4.6. Recall that a bounded function f is a function whose modulus $|f|$ is bounded, and an entire function is analytic over the entire complex plane. Thus a non-constant polynomial is entire, but it is not bounded.

Example 1. Suppose that $f(z)$ is entire and that $\Re f \le M$ for all z. If we consider

$$e^{f}=e^{\Re f}e^{i\Im f} \tag{1}$$

then, because $\Re f\le M$, $e^{\Re f}\le e^{M}$. Thus we have

$$\left|e^{f}\right|=\left|e^{\Re f}e^{i\Im f}\right|=e^{\Re f}\le e^{M} \tag{2}$$

Therefore e^{f} is bounded and entire, so must be a constant function, which in turn means that f itself is also a constant function.

Example 2. Let $f(z)$ be analytic and entire and suppose its fifth derivative $f^{(5)}(z)$ is bounded in the whole plane. From Liouville's theorem, we must have

$$f^{(5)}(z)=a_5 \tag{3}$$

for some constant a_5. Integrating this we have

$$f^{(4)}(z)=a_5z+a_4 \tag{4}$$

where a_4 is another constant. Continuing the same way, we find

$$
\begin{aligned}
f^{(3)}(z) &= \frac{a_5}{2} z^2 + a_4 z + a_3 \\
f^{(2)}(z) &= \frac{a_5}{2 \times 3} z^3 + \frac{a_4}{2} z^2 + a_3 z + a_2 \\
f'(z) &= \frac{a_5}{2 \times 3 \times 4} z^4 + \frac{a_4}{2 \times 3} z^3 + \frac{a_3}{2} z^2 + a_2 z + a_1 \\
f(z) &= \frac{a_5}{5!} z^5 + \frac{a_4}{4!} z^4 + \frac{a_3}{3!} z^3 + \frac{a_2}{2!} z^2 + a_1 z + a_0
\end{aligned} \tag{5}
$$

Thus $f(z)$ must be a polynomial of degree at most 5.