LOGARITHMS - A FEW PROPERTIES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 11 July 2021.

To treat the statistical physics of macroscopic objects, we need to deal with very large numbers of particles, typically on the order of 10^{23} or more. The methods we've used for analyzing interacting Einstein solids by calculating the number of microstates for each macrostate breaks down for such numbers, as computers aren't able to calculate the required binomial coefficients exactly.

One way of reducing a very large number to a smaller number is to take the logarithm, so we'll review a few properties of the logarithm here.

The natural logarithm, or logarithm to base e is defined so that

$$e^{\ln x} = x \tag{1}$$

The logarithm tends to $-\infty$ as $x \to 0$ and to $+\infty$ as $x \to +\infty$, although the latter limit is reached much more slowly than pretty well every other elementary function. The (real) logarithm is defined only for $x > 0$.

A plot is in Fig. 1.

A few identities involving the logarithm can be derived from its definition. First, the logarithm of a product:

$$e^{\ln ab} = ab = e^{\ln a} e^{\ln b} = e^{\ln a + \ln b} \tag{2}$$

$$\ln ab = \ln a + \ln b \tag{3}$$

For exponents:

$$e^{\ln a^b} = a^b = \left(e^{\ln a}\right)^b = e^{b \ln a} \tag{4}$$

$$\ln a^b = b \ln a \tag{5}$$

For the derivative, we can use implicit differentiation:

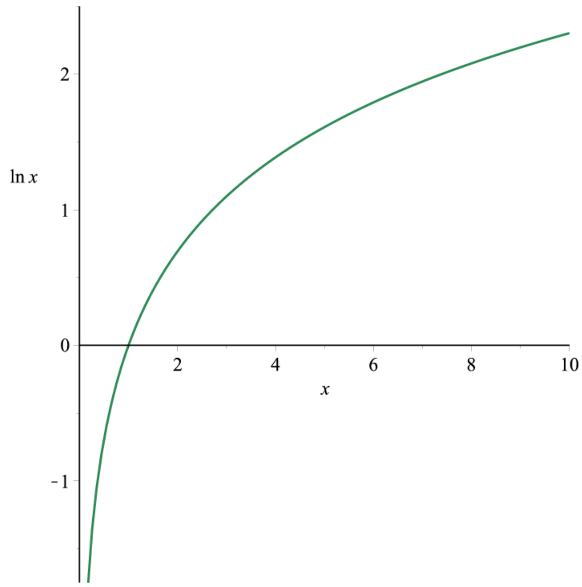

FIGURE 1. Plot of $\ln x$.

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x = 1 \tag{6}$$

$$e^{\ln x}\frac{d}{dx}(\ln x) = 1 \tag{7}$$

$$x\frac{d}{dx}(\ln x) = 1 \tag{8}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{9}$$

Since $\ln 1 = 0$ we can use a Taylor expansion to get an approximation for $\ln(1+x)$ for small x:

$$\ln(1+x) = \ln(1+0) + \left.\frac{1}{1+x}\right|_{x=0} x + \ldots \tag{10}$$

$$\approx 0 + x \tag{11}$$

$$= x \tag{12}$$

For $x = 0.1$, $\ln 1.1 = 0.0953$ which is fairly close to 0.1. For $x = 0.01$, $\ln 1.01 = 0.00995$ so the approximation is quite good here.

A more general form of this approximation can be derived for $\ln(a+b)$ where $b \ll a$. We get

$$\ln(a+b) = \ln\left[a\left(1+\frac{b}{a}\right)\right] \tag{13}$$

$$= \ln a + \ln\left(1+\frac{b}{a}\right) \tag{14}$$

$$\approx \ln a + \frac{b}{a} \tag{15}$$

The natural logarithm uses e as the base and is the most common logarithm in physics and mathematics because its properties are especially simple. Logarithms can be defined relative to any other real number, however, and the definition is similar to 1. Base 10 logs are defined so that

$$10^{\log x} = x \tag{16}$$

To convert to natural logs, take the natural log of both sides:

$$(\log x)(\ln 10) = \ln x \tag{17}$$

Thus

$$\log x = \frac{\ln x}{\ln 10} \tag{18}$$

and in general, the $\log_b$ to any base b is

$$\log_b x = \frac{\ln x}{\ln b} \tag{19}$$

A similar technique can convert bases of exponentiation as in

$$e^{10^{23}} = 10^x \tag{20}$$

$$10^{23} = x \ln 10 \tag{21}$$

$$x = 4.343 \times 10^{22} \tag{22}$$

PINGBACKS

Pingback: Einstein solids - multiplicity of large systems