MAPPING CIRCLES ONTO ELLIPSES

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Here's a quick example of mapping domains in the complex plane. We map the circle $|z|=\rho$ using the mapping

$$w=z+\frac{1}{z} \tag{1}$$

We can write the circle as

$$z=\rho e^{i\theta} \tag{2}$$

for $0\le\theta<2\pi$. Then using 1 we have

$$\begin{aligned}
w &= \rho e^{i\theta}+\frac{1}{\rho}e^{-i\theta} && (3)\\
&= \rho(\cos\theta+i\sin\theta)+\frac{1}{\rho}(\cos\theta-i\sin\theta) && (4)\\
&= \left(\rho+\frac{1}{\rho}\right)\cos\theta+i\left(\rho-\frac{1}{\rho}\right)\sin\theta && (5)
\end{aligned}$$

If we write w in terms of its real and imaginary parts as

$$w=u+iv \tag{6}$$

then

$$\begin{aligned}
u &= \left(\rho+\frac{1}{\rho}\right)\cos\theta\\
v &= \left(\rho-\frac{1}{\rho}\right)\sin\theta
\end{aligned} \tag{7}$$

from which we get

$$\frac{u^2}{\left(\rho+\frac{1}{\rho}\right)^2}+\frac{v^2}{\left(\rho-\frac{1}{\rho}\right)^2}=1 \tag{8}$$

which is the equation of an ellipse in the w plane, provided that $\rho \neq 1$. The semi-major axis is $\rho + \frac{1}{\rho}$ and the semi-minor axis is $\rho - \frac{1}{\rho}$.