MAXIMUM MODULUS PRINCIPLE

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A couple of useful theorems are given in Saff and Snider's book, Section 4.6.

Theorem 1. *If a function $f(z)$ is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D.*

Remember that a domain is an open connected set, which means that every point in a domain is an interior point. That is, a bounded domain does not include the boundary.

Theorem 2. *(Maximum modulus principle) A function $f(z)$ analytic in a* bounded *domain D that is continuous up to and including the boundary attains its maximum modulus on the boundary.*

At first glance, these two theorems might appear to contradict each other, but it must be noted that they apply to different situations. Theorem 1 applies only to functions that are defined on domains without being defined on any boundaries. Theorem 1 can thus also apply to entire functions, that is, functions that are analytic over the entire complex plane. Theorem 2 applies to bounded domains, and requires the function to be defined both within the domain and on the boundary.

Example 1. Let f and g be analytic in a bounded domain D and continuous up to and including the boundary B. Suppose that g is never zero and that

$$|f(z)| \leq |g(z)| \tag{1}$$

for all z on the boundary B. If we consider the function $q(z) = f(z)/g(z)$ then, because g is never zero, $q(z)$ is also analytic and continuous over D and the boundary B. Therefore $|f(z)/g(z)| = |f(z)|/|g(z)|$ must attain its maximum value on B. However, from 1 we must have

$$|q(z)| = \frac{|f(z)|}{|g(z)|} \leq 1 \tag{2}$$

on B. This means that the maximum value of $|q(z)|$ is 1 over the whole domain, so 1 must be true for all z in D as well.

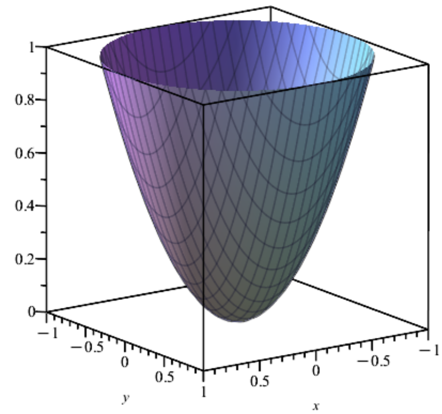

FIGURE 1. Plot of x^2+y^2 for $x\in[-1,1]$ and $y\in[-1,1]$.

Example 2. (Minimum modulus principle) Under the same conditions as Theorem 2 above with the additional requirement that f is not zero in the domain D, it is also true that $|f(z)|$ attains its *minimum* value on the boundary. To see this consider $1/f(z)$. Since f is not zero anywhere in D this function is also analytic and continuous up to and including the boundary, so $1/|f(z)|$ attains its maximum value on the boundary. Hence $|f(z)|$ must attain its minimum value on the boundary. Thus we have that an analytic function satisfying these conditions attains both its maximum and minimum values on the boundary.

To see that the requirement that $f(z)$ is non-zero everywhere in D, consider the domain $|z|<1$ with boundary B of $|z|=1$ and the function

$$f(z)=z^2=(x+iy)^2 \tag{3}$$

Note that $f(z)=0$ at $z=0$, and that

$$|f(z)|=x^2+y^2 \tag{4}$$

The function itself is analytic and continuous everywhere in D and on B, but the modulus attains its minimum at the origin, which is not on the boundary. See Fig. 1 which shows the minimum modulus in the interior of the domain.

If we restrict the domain so that it excludes the origin, then the maximum and minimum moduli are both on the boundary. Fig. 2 shows $|f|$ for a square domain excluding the origin. The minimum occurs at the lower left hand corner and the maximum at the upper right corner.

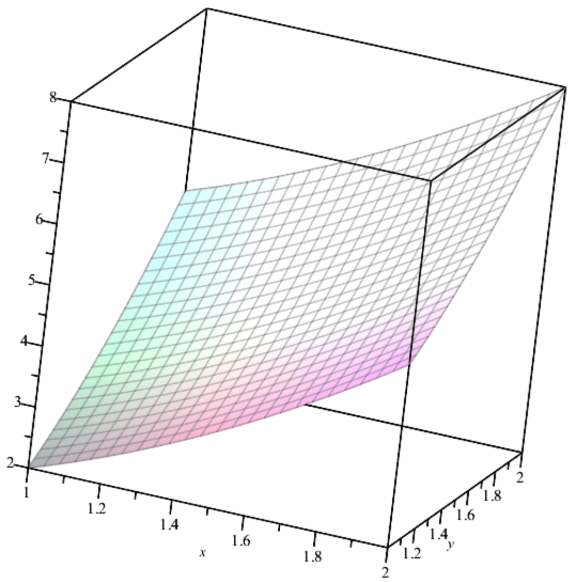

FIGURE 2. Plot of x^2+y^2 for $x \in [1,2]$ and $y \in [1,2]$.

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