MEROMORPHIC FUNCTIONS AND THE ARGUMENT PRINCIPLE

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Definition 1. A function is called *meromorphic* in a domain D if at every point in D it is either analytic or has a pole.

Example 1. The function

$$f(z) = 2z + z^3 \tag{1}$$

is meromorphic in the entire complex plane, as it is analytic everywhere and has no poles.

The function

$$f(z) = \frac{\sin z}{z^3 + 1} \tag{2}$$

is meromorphic over the entire plane, as it has simple poles at the three cube roots of -1 and is analytic everywhere else.

The function

$$f(z) = e^{1/z} \tag{3}$$

is not meromorphic, as it has an essential singularity (not a pole) at $z = 0$.

The function

$$f(z) = \tan z \tag{4}$$

is meromorphic as it is equal to $\sin z / \cos z$, and $\cos z$ has simple zeroes at $z = \left(n + \frac{1}{2}\right)\pi$, so $\tan z$ has simple poles at these points.

The function

$$f(z) = \frac{2i}{(z-3)^2} + \cos z \tag{5}$$

is meromorphic, as it has a pole of order 2 at $z = 3$, and $\cos z$ is analytic everywhere.

Theorem 1. *(Argument principle). Consider a function f that is analytic and nonzero on a simple, closed, positively oriented contour C, and is meromorphic inside C. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f) \tag{6}$$

where $N_0(f)$ is the number of zeroes (including multiplicity) of f inside C, and $N_p(f)$ is the number of poles (including multiplicity) of f inside C.

The proof is given in Saff and Snider, Section 6.7.

Example 2. Given a general polynomial of order n, with $a_n \neq 0$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \tag{7}$$

By the fundamental theorem of algebra, $P(z)$ has n zeroes. Some of the zeroes might be the same, so that the multiplicity of a given zero z_0 is ≥ 1. From the Argument Principle, we can say that, if we choose a contour of a circle of radius R, with R large enough so that the contour contains all the zeroes of P, then

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{P'(z)}{P(z)}dz = n \tag{8}$$

or

$$\int_{|z|=R} \frac{P'(z)}{P(z)}dz = 2n\pi i \tag{9}$$

Example 3. Given the function

$$f(z) = \frac{z^2 (z-i)^3 e^z}{3(z+2)^4 (3z-18)^5} \tag{10}$$

The function has an order 2 zero at $z = 0$ and an order 3 zero at $z = i$. Since e^z is never zero and is analytic everywhere, it doesn't enter into consideration. The function also a pole of order 4 at $z = -2$ and an order 5 pole at $z = 6$.

If we using the contour $|z| = 3$, the two zeroes and the pole at $z = -2$ are inside the circle, while the pole at $z = 6$ lies outside. Therefore

$$\begin{aligned} N_0(f) &= 2 + 3 = 5 \\ N_p(f) &= 4 \end{aligned} \tag{11}$$

$$\frac{1}{2\pi i}\int_{|z|=3}\frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f) \tag{12}$$
$$= 5-4=1 \tag{13}$$

[Remarkably, if we actually work out the derivative of 10 and use the parametrization

$$z = 3e^{it} \tag{14}$$

with $0 \le t \le 2\pi$ and get Maple to do the integral

$$\int_{|z|=3}\frac{f'(z)}{f(z)}dz = \int_0^{2\pi}\frac{f'\left(3e^{it}\right)}{f\left(3e^{it}\right)}\frac{dz}{dt}dt \tag{15}$$
$$= \int_0^{2\pi}\frac{f'\left(3e^{it}\right)}{f\left(3e^{it}\right)}3ie^{it}dt \tag{16}$$

it is capable of finding the answer of $2\pi i$. The actual intermediate expressions for $f\left(3e^{it}\right)$ and $f'\left(3e^{it}\right)$ are horrendous as you can imagine, so I won't state them here, but I was impressed.]

We can also work out the same integral around the circle $|z| = 7$, which includes all the zeroes and poles. Thus we get

$$\begin{aligned} N_0(f) &= 5 \\ N_p(f) &= 4+5=9 \end{aligned} \tag{17}$$

and

$$\frac{1}{2\pi i}\int_{|z|=7}\frac{f'(z)}{f(z)}dz = 5-9=-4 \tag{18}$$

This is also verified by Maple.