

NORMAL OPERATORS

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Hermitian and unitary operators are actually special cases of a more general type of operators known as *normal operators*. An operator is normal if it commutes with its adjoint:

$$[M^\dagger, M] = 0 \quad (1)$$

A hermitian operator is normal because $M^\dagger = M$. A unitary operator U is normal because $U^\dagger = U^{-1}$ and every operator commutes with its inverse.

Theorem 1. *A normal operator M remains normal after a similarity transformation with a unitary operator U .*

Proof. Suppose we transform M according to

$$M_1 = U^{-1}MU = U^\dagger MU \quad (2)$$

Then

$$[M_1^\dagger, M_1] = (U^\dagger M^\dagger U) (U^\dagger MU) - (U^\dagger MU) (U^\dagger M^\dagger U) \quad (3)$$

$$= U^\dagger M^\dagger MU - U^\dagger MM^\dagger U \quad (4)$$

$$= U^\dagger [M^\dagger, M] U \quad (5)$$

$$= 0 \quad (6)$$

Thus M_1 is normal. \square

Theorem 2. *If w is an eigenvector of the normal operator M with eigenvalue λ , it is also an eigenvector of the adjoint M^\dagger with eigenvalue λ^* .*

Proof. Suppose

$$Mw = \lambda w \quad (7)$$

$$(M - \lambda I)w = 0 \quad (8)$$

Consider the vector

$$u \equiv (M^\dagger - \lambda^* I) w \quad (9)$$

The norm of this vector satisfies

$$\langle u | u \rangle = \langle (M^\dagger - \lambda^* I) w | (M^\dagger - \lambda^* I) w \rangle \quad (10)$$

$$= \langle w | (M - \lambda I) (M^\dagger - \lambda^* I) w \rangle \quad (11)$$

$$= \langle w | (M^\dagger - \lambda^* I) (M - \lambda I) w \rangle \quad (12)$$

$$= 0 \quad (13)$$

where in the third line we used 1 and in the last line we used 8. Since the norm of u is 0, $u = 0$ and since $w \neq 0$, from 9 we have

$$(M^\dagger - \lambda^* I) w = 0 \quad (14)$$

$$M^\dagger w = \lambda^* w \quad (15)$$

as required. \square

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