NULL SPACE, RANGE, INJECTIVITY AND SURJECTIVITY

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We've looked at some basic properties of linear operators, so we'll carry on with a few more definitions and theorems.

1. Null space and injectivity

First, we define the *null space* or *kernel* of an operator T to be the set of all vectors which T maps to the zero vector:

$$\text{null}\,T=\{v\in V:Tv=0\} \tag{1}$$

Theorem 1. *The null space is a subspace.*

Proof. Since T is a linear operator, it maps the zero vector 0 to the zero vector, so $0\in\text{null}\,T$. If two other vectors $u,v\in\text{null}\,T$, then so is their sum, by the additivity property of linear operators. By the homogeneity property, if $u\in\text{null}\,T$ and $\lambda\in\mathbb{F}$, then $T(\lambda u)=\lambda Tu=0$. Thus null T is closed under addition and scalar multiplication, and contains the 0 vector, so it is a subspace. □

An operator T is *injective* if $Tu=Tv\rightarrow u=v$ where $\rightarrow$ means 'implies'. That is, no two different vectors are mapped to the same vector by the operator. An injective operator is also called *one-to-one* (or, in Zwiebach's notes, *two-to-two*).

A useful result is the following:

Theorem 2. *A linear operator T is injective if and only if null $T=\{0\}$. That is, if the only vector mapped to 0 is 0 itself, then T is one-to-one.*

Proof. If T is injective, then the only vector mapped to 0 is 0. To prove the converse, suppose null $T=\{0\}$ and assume there are two different vectors u,v such that $Tu=Tv$. Then $Tu-Tv=0=T(u-v)$. However, since null $T=\{0\}$ the only vector in the null space is 0, so we must have $u-v=0$ or $u=v$. Thus T is injective. □

2. Range and surjectivity

The *range* of an operator T is the set of all vectors produced by operating on vectors with T. That is

$$W = \text{range } T = \{Tv : v \in V\} \tag{2}$$

The notation $T \in \mathcal{L}(V,W)$ means the operator T operates on vector space V and has range W.

Theorem 3. *The range is a subspace.*

Proof. As before, we must have $T(0) = 0$, so the zero vector is in the range. To show that the range is closed under addition, choose two vectors $v_1, v_2 \in V$. The corresponding vectors in the range are $w_1 = Tv_1$ and $w_2 = Tv_2$. By linearity we must have $T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$, so W is closed under addition. Similarly for scalar multiplication, if $w = Tv$ then $T(\lambda v) = \lambda Tv = \lambda w$. □

An operator is surjective if $W = V$, that is, if the range is the same as the original vector space upon which T operates.

The null space and range of an operator obey the *fundamental theorem of linear maps*:

Theorem 4. *If V is finite-dimensional and $T \in \mathcal{L}(V,W)$ then range T is finite-dimensional and*

$$\textit{dim } V = \textit{dim null } T + \textit{dim range } T \tag{3}$$

That is, the dimension of the null space and the dimension of the range add up to the dimension of the original vector space on which T operates. Note that this makes sense only for finite-dimensional vector spaces.

Proof. Let $u_1, \ldots, u_m$ be a basis of null T. If dim null $T <$ dim V, we can add some more vectors to the basis $u_1, \ldots, u_m$ to get a basis for V (we haven't proved this, but the proof is in Axler section 2.33). Suppose we need another n vectors to do this. We then have a basis for V:

$$u_1, \ldots, u_m, v_1, \ldots, v_n \tag{4}$$

The dimension of V is therefore $m + n$. To complete the proof, we need to show that dim range $T = n$, which we can do if we can show that $Tv_1, \ldots, Tv_n$ is a basis for range T.

Since 4 is a basis for V, we can write any vector $v \in V$ as a linear combination

$$v=\sum_{i=1}^{m}a_iu_i+\sum_{j=1}^{n}b_iv_i \tag{5}$$

Operating on this equation with T and using the fact that $Tu_i=0$ since the u_i are a basis for the null space of T, we have

$$Tv=\sum_{j=1}^{n}b_iTv_i \tag{6}$$

Thus any vector in range T can be written as a linear combination of the vectors Tv_i, so the vectors Tv_i span the range of T.

To complete the proof, we need to show that the vectors Tv_i are linearly independent and thus form a basis for range T. To do this, suppose we have the equation

$$\sum_{i=1}^{n}c_iTv_i=0 \tag{7}$$

By linearity, we have

$$T\sum_{i=1}^{n}c_iv_i=0 \tag{8}$$

so $\sum_{i=1}^{n}c_iv_i\in$ null T and, since the u_i are a basis for null T, we can write

$$\sum_{i=1}^{n}c_iv_i=\sum_{i=1}^{m}d_iu_i \tag{9}$$

However, because the list 4 of vectors is a basis for V, all the vectors in this list are linearly independent, which means that the only way two different linear combinations of these vectors can be equal is if all the coefficients are zero, that is, all the c_is and d_is are zero. Going back to 7, this means that the only solution is $c_i=0$ for all i, which means that the vectors Tv_i are linearly independent and span range T, so they form a basis for range T. Thus dim range $T=n$. □

REFERENCES

(1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 1.
(2) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

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