ORTHONORMAL BASIS AND ORTHOGONAL COMPLEMENT

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 3 Jan 2021.

1. ORTHONORMAL BASIS

Once we have defined an inner product defined on a vector space V, we can create an orthonormal basis for V. A list of vectors $(e_1, e_2, \ldots, e_n)$ is orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij} \tag{1}$$

That is, any pair of vectors in the list are orthogonal, and all the vectors have norm 1. In 3-d space, the unit vectors along the three axes form an orthonormal list.

Given an orthonormal list, we can construct a vector from the vectors in that list by

$$\begin{aligned} v &= a_1 e_1 + a_2 e_2 + \ldots + a_n e_n && (2) \\ &= \sum_{i=1}^{n} a_i e_i && (3) \end{aligned}$$

for $a_i \in \mathbb{F}$. The norm of v has a simple form:

$$\begin{aligned} \langle v, v \rangle^2 &= \left\langle \sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i \right\rangle && (4) \\ &= \sum_{i=1}^{n} \langle a_i e_i, a_i e_i \rangle + \text{zero terms} && (5) \\ &= \sum_{i=1}^{n} |a_i|^2 && (6) \end{aligned}$$

The 'zero terms' in the second line are terms involving $\langle a_i e_i, a_j e_j \rangle$ for $i \neq j$ which are all zero because of 1.

This result shows that an orthonormal list of vectors is linearly independent, since if we form the linear combination

$$v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n = 0 \tag{7}$$

and require that this linear combination be zero, then $\langle v, v \rangle = 0$ so from 6 we must have all $a_i = 0$, which means the list is linearly independent.

If we have an orthonormal list $(e_1, e_2, \ldots, e_n)$ that is also a basis for V, then any vector $v \in V$ can be written as

$$v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n \tag{8}$$

The coefficients a_i can be found by taking the inner product $\langle e_i, v \rangle = a_i$ (using 1), so we have

$$v = \sum_{i=1}^{n} \langle e_i, v \rangle e_i \tag{9}$$

For example, in 3-d space, the 3 unit vectors along the x, y, z axes form an orthonormal basis for the space. However, the unit vectors along the x, y axes form an orthonormal list, but this is not a basis for 3-d space since no vector with a z component can be written as a linear combination of these two vectors.

2. GRAM-SCHMIDT ORTHOGONALIZATION

If we have any basis (not necessarily orthonormal), we can form an orthonormal basis using the Gram-Schmidt orthogonalization procedure. The procedure is iterative and follows these steps:

The first vector e_1 in the orthonormal basis is defined by

$$e_1 = \frac{v_1}{|v_1|} \tag{10}$$

where v_1 is the first vector (well, any vector, really) in the non-orthonormal basis. Given vector e_{j-1} in the orthonormal basis, we can form e_j from the formula

$$e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \tag{11}$$

We can check that $\langle e_k, e_j \rangle = \delta_{kj}$ by direct calculation. From its definition, e_j clearly has norm 1, so $\langle e_j, e_j \rangle = 1$. If $k \neq j$:

$$\langle e_k, e_j \rangle = \frac{\langle e_k, v_j \rangle - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle \langle e_k, e_i \rangle}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \tag{12}$$

$$= \frac{\langle e_k, v_j \rangle - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle \delta_{ki}}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \tag{13}$$

$$= \frac{\langle e_k, v_j \rangle - \langle e_k, v_j \rangle}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \tag{14}$$

$$= 0 \tag{15}$$

Note that although we've indexed the vectors v_i in the original basis, we can take them in any order when calculating the orthonormal basis via the Gram-Schmidt procedure.

3. Orthogonal complement

Suppose we have a subset U (not necessarily a subspace) of V. Then we can define the orthogonal complement $U^\perp$ of U as the set of all vectors that are orthogonal to all vectors $u \in U$. More formally:

$$U^\perp \equiv \{v \in V \,|\, \langle v, u \rangle = 0 \text{ for all } u \in U\} \tag{16}$$

A useful general theorem is as follows.

Theorem 1. *If U is a subspace of V, then $V = U \oplus U^\perp$. (Recall the direct sum.)*

Proof. Given an orthonormal basis of U: $(e_1, e_2, \ldots, e_n)$, we can write any $v \in V$ as an orthogonal decomposition

$$v = \underbrace{\sum_{i=1}^{n} \langle e_i, v \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^{n} \langle e_i, v \rangle e_i}_{\in U^\perp} \tag{17}$$

On the RHS, we've just added and subtracted the same term from v. Since the first term is a linear combination of the basis vectors of U, the overall sum is a vector in U. To see that the second term is in $U^\perp$, take the inner product with any of the basis vectors e_k:

$$\left\langle e_k, v - \sum_{i=1}^{n} \langle e_i, v \rangle e_i \right\rangle = \langle e_k, v \rangle - \left\langle e_k, \sum_{i=1}^{n} \langle e_i, v \rangle e_i \right\rangle \tag{18}$$

$$= \langle e_k, v \rangle - \sum_{i=1}^{n} \langle e_i, v \rangle \delta_{ik} \tag{19}$$

$$= \langle e_k, v \rangle - \langle e_k, v \rangle \tag{20}$$

$$= 0 \tag{21}$$

Finally, since the two vector spaces in a direct sum can have only the zero vector in their intersection, we need to show that $U \cap U^{\perp} = \{0\}$. However if a vector v is in both U and $U^{\perp}$ then it must be orthogonal to itself, so $\langle v, v \rangle = 0$ which implies $v = 0$. □

Thus any vector space V can be decomposed into two orthogonal subspaces (assuming that V has any subspaces other than $\{0\}$).

REFERENCES

(1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.
(2) Liesen, Jörg & Mehrmann, Volker (2015), *Linear Algebra,* Chapter 12.
(3) Landi, Giovanni & Zampini, Alessandro (2018), *Linear Algebra and Analytic Geometry for Physical Sciences*, Chapter 3.
(4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

PINGBACKS

Pingback: Linear functionals and adjoint operators
Pingback: Projection operators
Pingback: Spectral theorem for normal operators