

ORTHONORMAL BASIS AND ORTHOGONAL COMPLEMENT

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1. ORTHONORMAL BASIS

Once we have defined an inner product defined on a vector space V , we can create an orthonormal basis for V . A list of vectors (e_1, e_2, \dots, e_n) is orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij} \quad (1)$$

That is, any pair of vectors in the list are orthogonal, and all the vectors have norm 1. In 3-d space, the unit vectors along the three axes form an orthonormal list.

Given an orthonormal list, we can construct a vector from the vectors in that list by

$$v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n \quad (2)$$

$$= \sum_{i=1}^n a_i e_i \quad (3)$$

for $a_i \in \mathbb{F}$. The norm of v has a simple form:

$$\langle v, v \rangle^2 = \left\langle \sum_{i=1}^n a_i e_i, \sum_{i=1}^n a_i e_i \right\rangle \quad (4)$$

$$= \sum_{i=1}^n \langle a_i e_i, a_i e_i \rangle + \text{zero terms} \quad (5)$$

$$= \sum_{i=1}^n |a_i|^2 \quad (6)$$

The 'zero terms' in the second line are terms involving $\langle a_i e_i, a_j e_j \rangle$ for $i \neq j$ which are all zero because of 1.

This result shows that an orthonormal list of vectors is linearly independent, since if we form the linear combination

$$v = a_1e_1 + a_2e_2 + \dots + a_ne_n = 0 \quad (7)$$

and require that this linear combination be zero, then $\langle v, v \rangle = 0$ so from 6 we must have all $a_i = 0$, which means the list is linearly independent.

If we have an orthonormal list (e_1, e_2, \dots, e_n) that is also a basis for V , then any vector $v \in V$ can be written as

$$v = a_1e_1 + a_2e_2 + \dots + a_ne_n \quad (8)$$

The coefficients a_i can be found by taking the inner product $\langle e_i, v \rangle = a_i$ (using 1), so we have

$$v = \sum_{i=1}^n \langle e_i, v \rangle e_i \quad (9)$$

For example, in 3-d space, the 3 unit vectors along the x, y, z axes form an orthonormal basis for the space. However, the unit vectors along the x, y axes form an orthonormal list, but this is not a basis for 3-d space since no vector with a z component can be written as a linear combination of these two vectors.

2. GRAM-SCHMIDT ORTHOGONALIZATION

If we have any basis (not necessarily orthonormal), we can form an orthonormal basis using the Gram-Schmidt orthogonalization procedure. The procedure is iterative and follows these steps:

The first vector e_1 in the orthonormal basis is defined by

$$e_1 = \frac{v_1}{|v_1|} \quad (10)$$

where v_1 is the first vector (well, any vector, really) in the non-orthonormal basis. Given vector e_{j-1} in the orthonormal basis, we can form e_j from the formula

$$e_j = \frac{v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \quad (11)$$

We can check that $\langle e_k, e_j \rangle = \delta_{kj}$ by direct calculation. From its definition, e_j clearly has norm 1, so $\langle e_j, e_j \rangle = 1$. If $k \neq j$:

$$\langle e_k, e_j \rangle = \frac{\langle e_k, v_j \rangle - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle \langle e_k, e_i \rangle}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \quad (12)$$

$$= \frac{\langle e_k, v_j \rangle - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle \delta_{ki}}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \quad (13)$$

$$= \frac{\langle e_k, v_j \rangle - \langle e_k, v_j \rangle}{\left| v_j - \sum_{i=1}^{j-1} \langle e_i, v_j \rangle e_i \right|} \quad (14)$$

$$= 0 \quad (15)$$

Note that although we've indexed the vectors v_i in the original basis, we can take them in any order when calculating the orthonormal basis via the Gram-Schmidt procedure.

3. ORTHOGONAL COMPLEMENT

Suppose we have a subset U (not necessarily a subspace) of V . Then we can define the orthogonal complement U^\perp of U as the set of all vectors that are orthogonal to all vectors $u \in U$. More formally:

$$U^\perp \equiv \{v \in V \mid \langle v, u \rangle = 0 \text{ for all } u \in U\} \quad (16)$$

A useful general theorem is as follows.

Theorem 1. *If U is a subspace of V , then $V = U \oplus U^\perp$. (Recall the direct sum.)*

Proof. Given an orthonormal basis of U : (e_1, e_2, \dots, e_n) , we can write any $v \in V$ as an orthogonal decomposition

$$v = \underbrace{\sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U^\perp} \quad (17)$$

On the RHS, we've just added and subtracted the same term from v . Since the first term is a linear combination of the basis vectors of U , the overall sum is a vector in U . To see that the second term is in U^\perp , take the inner product with any of the basis vectors e_k :

$$\left\langle e_k, v - \sum_{i=1}^n \langle e_i, v \rangle e_i \right\rangle = \langle e_k, v \rangle - \left\langle e_k, \sum_{i=1}^n \langle e_i, v \rangle e_i \right\rangle \quad (18)$$

$$= \langle e_k, v \rangle - \sum_{i=1}^n \langle e_i, v \rangle \delta_{ik} \quad (19)$$

$$= \langle e_k, v \rangle - \langle e_k, v \rangle \quad (20)$$

$$= 0 \quad (21)$$

Finally, since the two vector spaces in a direct sum can have only the zero vector in their intersection, we need to show that $U \cap U^\perp = \{0\}$. However if a vector v is in both U and U^\perp then it must be orthogonal to itself, so $\langle v, v \rangle = 0$ which implies $v = 0$. \square

Thus any vector space V can be decomposed into two orthogonal subspaces (assuming that V has any subspaces other than $\{0\}$).

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