PARTIAL FRACTIONS - EXAMPLES

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A rational function of two polynomials of the form

$$R_{m,n}(z)=\frac{a_0+a_1z+a_2z^2+\ldots+a_mz^m}{b_n(z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\ldots(z-\zeta_r)^{d_r}} \tag{1}$$

can be written as a partial fraction decomposition as

$$\begin{aligned}R_{m,n}(z)=&\frac{A_0^{(1)}}{(z-\zeta_1)^{d_1}}+\frac{A_1^{(1)}}{(z-\zeta_1)^{d_1-1}}+\ldots+\frac{A_{d_1-1}^{(1)}}{z-\zeta_1}+\\&\frac{A_0^{(2)}}{(z-\zeta_2)^{d_2}}+\ldots+\frac{A_{d_2-1}^{(2)}}{z-\zeta_2}+\ldots+\\&\frac{A_0^{(r)}}{(z-\zeta_r)^{d_r}}+\ldots+\frac{A_{d_r-1}^{(r)}}{z-\zeta_r}\end{aligned} \tag{2}$$

where

$$A_s^{(j)}=\lim_{z\to\zeta_j}\frac{1}{s!}\frac{d^s}{dz^s}\left[(z-\zeta_j)^{d_j}R_{m,n}(z)\right] \tag{3}$$

Here are a few examples.

Example 1. $R_{0,3}=\frac{3+i}{z(z+1)(z+2)}$. In this example, the denominator is already factored, so we can apply 3 directly. We have

$$R_{0,3}=\frac{A_0^{(1)}}{z}+\frac{A_0^{(2)}}{z+1}+\frac{A_0^{(3)}}{z+2} \tag{4}$$

with

$$A_0^{(1)} = \lim_{z\to 0} zR_{0,3} = \frac{3+i}{2} \tag{5}$$
$$A_0^{(2)} = \lim_{z\to -1} (z+1)\,R_{0,3} = -3-i \tag{6}$$
$$A_0^{(3)} = \lim_{z\to -2} (z+2)\,R_{0,3} = \frac{3+i}{2} \tag{7}$$

so

$$R_{0,3} = \frac{(3+i)/2}{z} - \frac{3+i}{z+1} + \frac{(3+i)/2}{z+2} \tag{8}$$

Example 2. $R_{1,3} = \frac{2z+i}{z^3+z}$. In this case, we need to factor the denominator first. We have

$$z^3+z = z\left(z^2+1\right) \tag{9}$$
$$= z\,(z+i)\,(z-i) \tag{10}$$

Using the same technique as in Example 1, we get

$$R_{1,3} = \frac{A_0^{(1)}}{z} + \frac{A_0^{(2)}}{z+i} + \frac{A_0^{(3)}}{z-i} \tag{11}$$
$$A_0^{(1)} = \lim_{z\to 0} zR_{1,3} = i \tag{12}$$
$$A_0^{(2)} = \lim_{z\to -i} (z+i)\,R_{1,3} = \frac{i}{2} \tag{13}$$
$$A_0^{(3)} = \lim_{z\to i} (z-i)\,R_{1,3} = -\frac{3i}{2} \tag{14}$$

so

$$R_{1,3} = \frac{i}{z} + \frac{i/2}{z+i} - \frac{3i/2}{z-i} \tag{15}$$

Example 3. $R_{1,4} = \frac{z}{\left(z^2+z+1\right)^2}$. Again, we need to factor the denominator. We can use the quadratic formula to do this, with the result

$$z^2+z+1 = \left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right) \tag{16}$$

The partial fraction decomposition is therefore

$$R_{1,4} = \frac{A_0^{(1)}}{\left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2} + \frac{A_1^{(1)}}{\left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)} + \frac{A_0^{(2)}}{\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^2} + \frac{A_1^{(2)}}{\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)} \tag{17}$$

We can now apply 3. We have (using Maple to simplify the answers)

$$A_0^{(1)} = \lim_{z\to -\frac{1}{2}-\frac{\sqrt{3}i}{2}} \left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2 R_{1,4} = \frac{1}{6}+\frac{\sqrt{3}i}{6} \tag{18}$$

$$A_1^{(1)} = \lim_{z\to -\frac{1}{2}-\frac{\sqrt{3}i}{2}} \frac{d}{dz}\left[\left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2 R_{1,4}\right] = -\frac{\sqrt{3}i}{9} \tag{19}$$

$$A_0^{(2)} = \lim_{z\to -\frac{1}{2}+\frac{\sqrt{3}i}{2}} \left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^2 R_{1,4} = \frac{1}{6}-\frac{\sqrt{3}i}{6} \tag{20}$$

$$A_1^{(2)} = \lim_{z\to -\frac{1}{2}+\frac{\sqrt{3}i}{2}} \frac{d}{dz}\left[\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^2 R_{1,4}\right] = \frac{\sqrt{3}i}{9} \tag{21}$$

Thus we get

$$R_{1,4} = \frac{\left(1+\sqrt{3}i\right)/6}{\left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2} - \frac{\sqrt{3}i/9}{\left(z+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)} + \frac{\left(1-\sqrt{3}i\right)/6}{\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^2} + \frac{\sqrt{3}i/9}{\left(z+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)} \tag{22}$$

Example 4. $p(z) = \frac{5z^4+3z^2+1}{2z^2+3z+1}$. This time, the numerator has a higher degree than the denominator, so we need to do some long division first. We have

$$\frac{5z^4+3z^2+1}{2z^2+3z+1} = \frac{5}{2}z^2 - \frac{15}{4}z + \frac{47}{8} - \frac{(111z+39)/8}{2z^2+3z+1} \tag{23}$$

The partial fraction decomposition applies to the last (remainder) term, where the denominator factors, so we have

$$-\frac{(111z+39)/8}{2z^2+3z+1} = -\frac{(111z+39)/8}{(2z+1)(z+1)} \tag{24}$$

Applying the method above (I won't give the details, since it's essentially the same technique as in Example 1), we have

$$-\frac{(111z+39)/8}{(2z+1)(z+1)} = \frac{33/8}{2z+1} - \frac{9}{z+1} \tag{25}$$

so we have

$$\frac{5z^4+3z^2+1}{2z^2+3z+1}=\frac{5}{2}z^2-\frac{15}{4}z+\frac{47}{8}+\frac{33/8}{2z+1}-\frac{9}{z+1} \tag{26}$$