PARTIAL FRACTIONS

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You may have encountered partial fractions as a technique in an introductory calculus course, where it is used as a tool in evaluating some integrals. Its use is actually somewhat more widespread, so we present the basic theorem here, along with an example of how to calculate a partial fraction decomposition.

The main theorem is:

Theorem 1. *Given a rational function $R_{m,n}(z)$ of two polynomials:*

$$R_{m,n}(z)=\frac{a_0+a_1z+a_2z^2+\ldots+a_mz^m}{b_n(z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\ldots(z-\zeta_r)^{d_r}} \tag{1}$$

Here the degree m of the numerator is assumed to be less than the degree $n=\sum_{k=1}^{r}d_k$ of the denominator. Then this rational function has a partial fraction decomposition given by

$$\begin{aligned}R_{m,n}(z)=&\frac{A_0^{(1)}}{(z-\zeta_1)^{d_1}}+\frac{A_1^{(1)}}{(z-\zeta_1)^{d_1-1}}+\ldots+\frac{A_{d_1-1}^{(1)}}{z-\zeta_1}+\\&\frac{A_0^{(2)}}{(z-\zeta_2)^{d_2}}+\ldots+\frac{A_{d_2-1}^{(2)}}{z-\zeta_2}+\ldots+\\&\frac{A_0^{(r)}}{(z-\zeta_r)^{d_r}}+\ldots+\frac{A_{d_r-1}^{(r)}}{z-\zeta_r}\end{aligned} \tag{2}$$

where the $A_s^{(j)}$ are constants. It's useful to be clear about the various symbols here. In $A_s^{(j)}$ the j refers to the j in ζ_j, the jth root in the denominator. The s refers to the power to which the denominator is raised, in the form: for term $A_s^{(j)}$, it is associated with the denominator $(z-\zeta_j)^{d_1-s}$.

The proof of this theorem is rather involved and doesn't add anything to our understanding of how the partial fractions are calculated, so we won't give it here. If you're interested, the proof is given in Section 3.1 of Saff and Snider's book.

As with the Taylor form of a polynomial, there is a brute force method and an easier method for finding partial fractions. The brute force method (which is probably what you were taught in a calculus course) involves putting all the terms in 2 over a common denominator and then comparing the coefficients of each power of z in the resulting numerator with 1. In a simple case, this method works fairly well, but in more complicated cases we are faced with a large system of linear equations to solve, which can get messy.

A simpler method is as follows.

Example 1. Find partial fractions for

$$R_{1,4}(z)=\frac{4z+4}{z(z-1)(z-2)^3} \tag{3}$$

From 2 we see that the partial fraction form will have 5 terms:

$$R_{1,4}(z)=\frac{A_0^{(1)}}{z}+\frac{A_0^{(2)}}{z-1}+\frac{A_0^{(3)}}{(z-2)^3}+\frac{A_1^{(3)}}{(z-2)^2}+\frac{A_2^{(3)}}{(z-2)} \tag{4}$$

To find $A_0^{(1)}$, we note that if we multiply 4 through by z and then set $z=0$, we eliminate all terms except the first. That is

$$zR_{1,4}(z)=A_0^{(1)}+z\left[\frac{A_0^{(2)}}{z-1}+\frac{A_0^{(3)}}{(z-2)^3}+\frac{A_1^{(3)}}{(z-2)^2}+\frac{A_2^{(3)}}{(z-2)}\right] \tag{5}$$

Thus at $z=0$, we find

$$A_0^{(1)}=\left.\frac{4z+4}{(z-1)(z-2)^3}\right|_{z=0}=\frac{1}{2} \tag{6}$$

Similarly, to find $A_0^{(2)}$, we multiply through by $z-1$ and take $z=1$:

$$A_0^{(2)}=\left.\frac{4z+4}{z(z-2)^3}\right|_{z=1}=-8 \tag{7}$$

To find the three $A_s^{(3)}$ terms, we first multiply 4 through by $(z-2)^3$. This isolates the $A_0^{(3)}$ term, and leaves at least a factor of $(z-2)$ on all the other terms, so we then set $z=2$ to find

$$A_0^{(3)}=\left.\frac{4z+4}{z(z-1)}\right|_{z=2}=6 \tag{8}$$

To find $A_1^{(3)}$, we again start by multiplying through by $(z-2)^3$ which gives, from 4

$$(z-2)^3 R_{1,4}(z) = (z-2)^3 \left[\frac{A_0^{(1)}}{z} + \frac{A_0^{(2)}}{z-1}\right] + A_0^{(3)} + (z-2) A_1^{(3)} + (z-2)^2 A_2^{(3)} \tag{9}$$

If we now take the derivative of this with respect to z, the $A_0^{(3)}$ term vanishes, and all the other terms except $A_1^{(3)}$ will have at least one factor of $(z-2)$ remaining. If we now set $z=2$ in the derivative, we will thus find $A_1^{(3)}$. That is

$$A_1^{(3)} = \frac{d}{dz}\left((z-2)^3 R_{1,4}(z)\right)\Big|_{z=2} \tag{10}$$

We find (using Maple for the derivative, or do it by hand using the quotient rule):

$$A_1^{(3)} = \frac{d}{dz}\left(\frac{4z+4}{z(z-1)}\right)\Big|_{z=2} \tag{11}$$

$$= \frac{4}{z(z-1)} - \frac{4z+4}{z^2(z-1)} - \frac{4z+4}{z(z-1)^2}\Big|_{z=2} \tag{12}$$

$$= -7 \tag{13}$$

Finally, to find $A_2^{(3)}$, we start from 9 and take the second derivative and set $z=2$. This eliminates all terms except for $A_2^{(3)}$ and we have

$$\frac{d^2}{dz^2}\left((z-2)^3 R_{1,4}(z)\right)\Big|_{z=2} = 2A_2^{(3)} \tag{14}$$

[The factor of 2 on the RHS comes from the second derivative of $(z-2)^2$ in the last term of 9.]

Finding this second derivative (again using Maple, as it gets a bit messy):

$$\frac{d^2}{dz^2}\left((z-2)^3 R_{1,4}(z)\right) = -\frac{8}{z^2(z-1)} - \frac{8}{z(z-1)^2} + \frac{2(4z+4)}{z^3(z-1)} + \frac{2(4z+4)}{z^2(z-1)^2} + \frac{2(4z+4)}{z(z-1)^3} \tag{15}$$

Substituting $z=2$ we get

$$2A_2^{(3)} = 15 \tag{16}$$

$$A_2^{(3)} = \frac{15}{2} \tag{17}$$

Thus the partial fraction decomposition is

$$R_{1,4}(z) = \frac{1}{2z} - \frac{8}{z-1} + \frac{6}{(z-2)^3} - \frac{7}{(z-2)^2} + \frac{15}{2(z-2)} \tag{18}$$

The method can be generalized to get the formula

$$\boxed{A_s^{(j)} = \lim_{z\to\zeta_j} \frac{1}{s!}\frac{d^s}{dz^s}\left[(z-\zeta_j)^{d_j} R_{m,n}(z)\right]} \tag{19}$$

The $\frac{1}{s!}$ is required to cancel out the $s!$ that is generated by taking the sth derivative of $(z-\zeta_j)^{d_j}$. Remember that $0! = 1$ so this formula works even when $s = 0$.

Admittedly, taking higher order derivatives of rational functions can get somewhat messy (as we've seen here), but it's probably still simpler than using the brute force method of solving a large set of linear equations.

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