POINT AT INFINITY

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In real variables, there are two infinities: $+\infty$ and $-\infty$. When we use complex numbers, we can go off to infinity in any direction, say, along the real axes, or along the imaginary axes, or along any other direction out from the origin. In practice, the *point at infinity* in the complex plane is considered to be a single point, and the incorporation of infinity into the complex plane gives what is known as the *extended complex plane*. The basis for considering the point at infinity to be a single point is that all points infinitely far from the origin map onto the single point (the north pole) on the Riemann sphere.

Just as with other points, a complex function f can have zeroes or singularities at infinity. There are various ways of finding these but probably the easiest is as follows. Given a function $f(z)$, if we want to classify its behaviour at infinity, we examine the function $f(1/w)$. Then

- If $f(1/w)$ is analytic or has a removable singularity at $w=0$, then $f(z)$ is analytic at $z\to\infty$.
- If $f(1/w)$ has a pole of order m at $w=0$, then $f(z)$ has a pole of order m at $z\to\infty$.
- If $f(1/w)$ has an essential singularity at $w=0$, then $f(z)$ has an essential singularity at $z\to\infty$.

Here are a few examples.

Example 1. $f(z)=e^{z}$. At first glance, we would be tempted to say that $e^{z}\to\infty$ as $z\to\infty$, but that assumes that z is real. However, using the guidelines above, we investigate $e^{1/w}$ near $w=0$. We've seen before that $e^{1/w}$ has an essential singularity at $w=0$, since values taken on by $e^{1/w}$ depend on the path by which we approach zero. As we approach $w=0$ along the positive real axis, $e^{1/w}\to\infty$, along the negative real axis, $e^{1/w}\to-\infty$, and along the imaginary axes, the function oscillates and always has modulus 1. Thus e^{z} has an essential singularity at ∞.

We can also see this from the series expansion.

$$e^{1/w}=\sum_{k=0}^{\infty}\frac{1}{k!w^{k}} \tag{1}$$

Thus there are an infinite number of terms with negative exponents, which indicates an essential singularity.

Example 2. $f(z) = \cosh z$. The easiest way to see this is by the series expansion.

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \tag{2}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \tag{3}$$

For $\cosh \frac{1}{w}$ we have

$$\cosh \frac{1}{w} = \sum_{k=0}^{\infty} \frac{1}{(2k)! w^{2k}} \tag{4}$$

As with the exponential, there are an infinite number of terms with negative exponents, so $\cosh z$ has an essential singularity at ∞.[1]

Example 3. $f(z) = \frac{z-1}{z+1}$. We have

$$f\left(\frac{1}{w}\right) = \frac{1/w - 1}{1/w + 1} \tag{5}$$

$$= \frac{1-w}{1+w} \tag{6}$$

As $w \to 0$, $f\left(\frac{1}{w}\right) \to 1$, so $f\left(\frac{1}{w}\right)$ has a removable singularity at $w = 0$. Thus $f(z)$ is analytic at ∞.

Example 4. $f(z) = \frac{z}{z^3+i}$. We have

$$f\left(\frac{1}{w}\right) = \frac{1/w}{1/w^3 + i} \tag{7}$$

$$= \frac{w^2}{1 + iw^3} \tag{8}$$

[1]For anyone tempted to use AI tools to solve these problems, beware! I fed this problem into ChatGPT and it came up with the answer that $\cosh z$ has a simple pole at ∞. I then asked it to prove that $\cosh z$ had an essential singularity at ∞, and it produced a proof of that too. Thus it came up with two contradictory answers to the same problem. AI still has a way to go...

As $w \to 0$, $f\left(\frac{1}{w}\right) \to 0$, so it has a removable singularity at $w = 0$. Thus $f(z)$ is analytic at ∞.

Example 5. $f(z) = \frac{z^3+i}{z}$. We have

$$f\left(\frac{1}{w}\right) = \frac{1/w^3 + i}{1/w} \tag{9}$$

$$= \frac{1 + iw^3}{w^2} \tag{10}$$

This has a pole of order 2 at $w = 0$, so $f(z)$ has a pole of order 2 at ∞.

Example 6. $f(z) = e^{\sinh z}$. Using the series expansion of $\sinh z$, we have

$$\sinh z = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{11}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \tag{12}$$

Thus

$$\sinh \frac{1}{w} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!w^{2k+1}} \tag{13}$$

As there are an infinite number of terms with negative exponents, $\sinh \frac{1}{w}$ has an essential singularity at $w = 0$. By Picard's theorem, this means that $\sinh \frac{1}{w}$ takes on (almost) every complex number in every neighbourhood of $w = 0$. Thus $e^{\sinh z}$ also takes on (almost) every complex number in every neighbourhood of $z \to \infty$ and it has an essential singularity there.

Example 7. $f(z) = \frac{\sin z}{z^2}$. From the series expansion, we have

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) \tag{14}$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots \tag{15}$$

Thus

$$f\left(\frac{1}{w}\right) = w - \frac{1}{3!w} + \frac{1}{5!w^3} - \ldots \tag{16}$$

As there are an infinite number of terms with negative exponents, $f\left(\frac{1}{w}\right)$ has an essential singularity at $w = 0$, so $f(z)$ has an essential singularity at ∞. This is counter-intuitive at first glance, as we're used to thinking of $\sin z$ as bounded within the interval $[-1, 1]$, but the sine of a complex number can take on any value due to its definition as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{17}$$

Example 8. $f(z) = \frac{1}{\sin z}$. As we've just seen in the previous example, $\sin z$ has an essential singularity at ∞, so this function does too. In fact, all the basic trig functions have an essential singularity at infinity. Thus an entire function (one that is analytic over the entire complex plane) may actually have a singularity in the extended complex plane.

Example 9. $f(z) = e^{\tan 1/z}$. As $z \to \infty$, $\frac{1}{z} \to 0$, $\tan 1/z \to 0$, so $e^{\tan 1/z} \to 1$. Thus $e^{\tan 1/z}$ is analytic at ∞.