POINTWISE AND UNIFORM CONVERGENCE

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Definition 1. A sequence $\{F_n(z)\}_{n=1}^{\infty}$ is said to converge *pointwise* to a function $F(z)$ on the set T if for any $\varepsilon > 0$ there exists an integer N_z such that when $n > N_z$

$$|F(z) - F_n(z)| < \varepsilon \tag{1}$$

However, for pointwise convergence, the value of N_z may depend on z. That is, we may have to calculate more terms in the sequence in order for 1 to be satisfied at some values of z than at other values.

Definition 2. A sequence $\{F_n(z)\}_{n=1}^{\infty}$ is said to *uniformly* converge to a function $F(z)$ on the set T if for any $\varepsilon > 0$ there exists an integer N such that when $n > N$

$$|F(z) - F_n(z)| < \varepsilon \tag{2}$$

for *all* $z \in T$. Note that uniform convergence is a more stringent condition than pointwise convergence, as it requires that there is a finite value of N for which 2 is satisfied for *all* points in the set T. A sequence that is uniformly convergent is also pointwise convergent.

Example 1. Consider the sequence of functions defined on the open interval $T = (0,1)$:

$$F_n(x) = x^n \qquad (n = 0, 1, 2, \ldots) \tag{3}$$

and take $\varepsilon = \frac{1}{2}$. We consider the convergence of this sequence to the function $F(x) = 0$ on T. That is, we require $x^n < \frac{1}{2}$ for $n > N$. Taking logs, we have

$$n\text{Log}\, x < \text{Log}\, \frac{1}{2} = -\text{Log}\, 2 \tag{4}$$

Taking the minus sign to the other side (and thus reversing the inequality) we have

$$n(-\text{Log}\, x) = n\text{Log}\, x^{-1} > \text{Log}\, 2 \tag{5}$$

Thus the minimum value of N is N_x where

$$N_x > \frac{\text{Log } 2}{\text{Log } x^{-1}} \tag{6}$$

[Note that $x^{-1} > 1$, so Log $x^{-1} > 0$, so we don't need to reverse the inequality when dividing by Log x^{-1}.]

Thus in this case, N_x always depends on x and thus this sequence is not uniformly convergent. It *is* pointwise convergent, since for any $x \in (0,1)$ it is possible to make $x^n < \frac{1}{2}$ if we take n large enough. However, for values of x very close to 1, the value of n can increase without limit.

Example 2. Is the geometric series uniformly convergent on the set $|z| < 1$? The series is

$$S = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \tag{7}$$

For $|z| < 1$, the RHS is finite, so the series converges pointwise on this set. However, for uniform convergence, we would require the partial sum

$$S_n = \sum_{k=0}^{n} z^k \tag{8}$$

to converge to $1/(1-z)$ for all z, whenever n is greater than some finite value N. That is, we need to look at

$$|S_n - S| = \left| \frac{1-z^{n+1}}{1-z} - \frac{1}{1-z} \right| \tag{9}$$

$$= \left| \frac{z^{n+1}}{1-z} \right| < \varepsilon \tag{10}$$

We want to find n such that

$$\frac{|z|^{n+1}}{|1-z|} < \varepsilon \tag{11}$$

Taking logs,

$$(n+1)\,\text{Log } |z| < \text{Log } \varepsilon + \text{Log } |1-z| \tag{12}$$

Since $|z| < 1$, Log $|z| < 0$, so

$$n > \frac{\text{Log } \varepsilon + \text{Log } |1-z|}{\text{Log } |z|} - 1 \tag{13}$$

All three of the terms in the quotient on the RHS are logs of numbers less than 1 (we're assuming ε is very small), so they are all negative, so the quotient is positive. As $|z| \to 1$ from below, Log $|z| \to 0$, Log $|1-z| \to -\infty$ and Log ε is a negative constant. Thus the value of n required to satisfy 10 increases without limit as $|z| \to 1$. Therefore the required value of n depends on z, and the sequence of partial sums of the geometric series is not uniformly convergent.

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