POISSON INTEGRAL FORMULA FOR THE HALF-PLANE

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The Poisson integral formula can also be applied to the upper half-plane.

Consider the contour Γ_R shown in Fig. 1, which consists of a line segment along the real axis from $-R$ to R and a semi-circular arc C_R^+ connecting the ends of the line segment. We also label an arbitrary point z within the contour. We consider a function

$$f = \phi + i\psi \tag{1}$$

which is analytic in a domain containing the real axis and the upper half-plane. The modulus $|f(z)| \le K$ within this domain.

From the Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(\zeta)}{\zeta - z}d\zeta \tag{2}$$

Since the complex conjugate $\bar{z}$ lies outside the contour Γ_R, the function $f(\zeta)/(\zeta - \bar{z})$ is analytic within the contour, so Cauchy's integral theorem tells us that its integral around Γ_R must vanish. That is

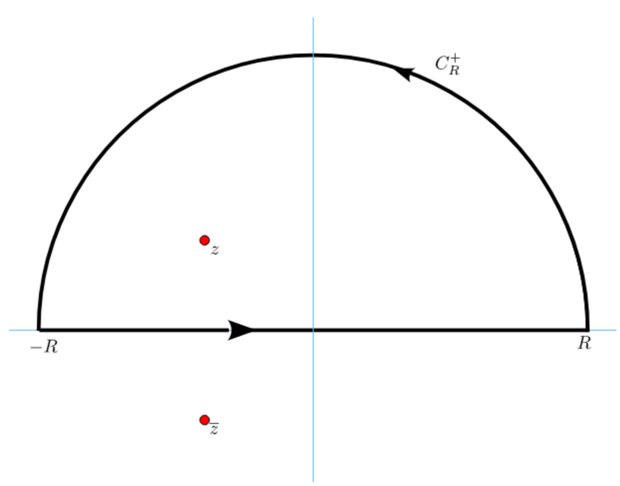

FIGURE 1. Contour for Poisson integral formula.

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(\zeta)}{\zeta-\overline{z}}d\zeta=0 \tag{3}$$

Taking the difference of 2 minus 3 we have

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(\zeta)}{\zeta-\overline{z}}d\zeta \tag{4}$$

We can consider the contour integral in two parts. Along the real axis, we have

$$\frac{1}{2\pi i}\int_{-R}^{R}f(\xi)\left[\frac{1}{\xi-z}-\frac{1}{\xi-\overline{z}}\right]d\xi \tag{5}$$

We've called the integration variable ξ to emphasize the fact that it is a real variable, since the integral is along the real axis. Therefore the two terms in the brackets are complex conjugates of each other. Putting over a common denominator, we have

$$\frac{1}{\xi-z}-\frac{1}{\xi-\overline{z}}=\frac{\xi-\overline{z}-\xi+z}{(\xi-z)(\xi-\overline{z})} \tag{6}$$

$$=\frac{2i\Im z}{|\xi-z|^2} \tag{7}$$

Thus

$$\frac{1}{2\pi i}\int_{-R}^{R}f(\xi)\left[\frac{1}{\xi-z}-\frac{1}{\xi-\overline{z}}\right]d\xi=\frac{1}{2\pi i}\int_{-R}^{R}f(\xi)\frac{2i\Im z}{|\xi-z|^2}d\xi \tag{8}$$

Along C_R^+, the integration variable remains complex so we have

$$\frac{1}{\zeta-z}-\frac{1}{\zeta-\overline{z}}=\frac{\zeta-\overline{z}-\zeta+z}{(\zeta-z)(\zeta-\overline{z})}=\frac{2i\Im z}{(\zeta-z)(\zeta-\overline{z})} \tag{9}$$

so

$$\frac{1}{2\pi i}\int_{C_R^+}f(\zeta)\left[\frac{1}{\zeta-z}-\frac{1}{\zeta-\overline{z}}\right]d\zeta=\frac{1}{2\pi i}\int_{C_R^+}f(\zeta)\frac{2i\Im z}{(\zeta-z)(\zeta-\overline{z})}d\zeta \tag{10}$$

Therefore

$$f(z)=\frac{1}{2\pi i}\int_{-R}^{R}f(\xi)\frac{2i\Im z}{|\xi-z|^2}d\xi+\frac{1}{2\pi i}\int_{C_R^+}f(\zeta)\frac{2i\Im z}{(\zeta-z)(\zeta-\overline{z})}d\zeta \tag{11}$$

We can now get an upper bound on the second integral (the one over C_R^+). To do this, we use the assumption that $|f(z)|\le K$, but we also need lower bounds on the terms $|\zeta - z|$ and $|\zeta - \overline{z}|$ in the denominator. From Fig. 1, we see that the lower bound on $|\zeta - z|$ is the minimum distance from z to the semicircular arc, which gives us

$$|\zeta - z| \geq R - |z| \tag{12}$$

Since $\overline{z}$ lies outside the contour the distance $|\zeta - \overline{z}|$ is always greater than $|\zeta - z|$, so we can also state that

$$|\zeta - \overline{z}| \geq R - |z| \tag{13}$$

Finally, since we're considering only points z in the upper half-plane,

$$\Im z \geq 0 \tag{14}$$

Putting it all together, we have, since the length of C_R^+ is πR:

$$\left| \frac{1}{2\pi i} \int_{C_R^+} f(\zeta) \frac{2i\Im z}{(\zeta - z)(\zeta - \overline{z})} d\zeta \right| \leq \frac{1}{2\pi} \frac{2K\Im z}{(R - |z|)^2} \pi R \tag{15}$$

$$= \frac{KR\Im z}{(R - |z|)^2} \tag{16}$$

As $R \to \infty$, this term goes to zero at order $1/R$, so as we expand the arc to infinity, this integral goes to zero. Therefore we have

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{\Im z}{|\xi - z|^2} d\xi \tag{17}$$

To get an expression for the real part ϕ in 1, we take the real part of this equation. Letting $z = x + iy$, we have

$$|\xi - z|^2 = (\xi - x)^2 + y^2 \tag{18}$$

$$\Im z = y \tag{19}$$

so we get, for $y > 0$

$$\boxed{\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(\xi, 0)}{(\xi - x)^2 + y^2} d\xi} \tag{20}$$

This gives the values of the harmonic function $\phi(x, y)$ in terms of its values along the real axis.

Poisson integral formula for the half-plane

Just as with the Poisson integral formula for circles, there is a more general form in which the values along the real axis can have a finite number of jump discontinuities. In this case, the formula is valid for either half-plane (positive y or negative y):

$$\boxed{\phi(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{U(\xi)}{(\xi-x)^2+y^2}d\xi} \tag{21}$$

Example 1. Suppose the temperature along the real axis is maintained at $T=1$ for $-1\le\xi\le 1$ and $T=0$ everywhere else on the real axis. We can use 21 to find the temperature everywhere on the upper half-plane. We have

$$\phi(x,y)=\frac{y}{\pi}\int_{-1}^{1}\frac{1}{(\xi-x)^2+y^2}d\xi \tag{22}$$

The integral can be done by substitution, though I used Maple to get

$$\phi(x,y)=\frac{1}{\pi}\left[\arctan\left(\frac{x+1}{y}\right)-\arctan\left(\frac{x-1}{y}\right)\right] \tag{23}$$