POISSON INTEGRAL FORMULA

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The Poisson integral formula applies to a harmonic function defined over a circular domain and its boundary. It is stated in the theorem:

Theorem 1. *Let U be a real-valued function defined on the circle C_R : $|z| = R$ and continuous on this circle except for a finite number of jump discontinuities. Then the function*

$$u\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{U\left(Re^{it}\right)}{R^2 + r^2 - 2rR\cos(t-\theta)} dt \tag{1}$$

is harmonic inside C_R. Further, as $re^{i\theta}$ approaches any point on C_R where U is continuous, $u\left(re^{i\theta}\right)$ approaches the value of U at that point.

Saff and Snider prove, in section 4.7, a restricted version of this theorem where U is continuous over all of C_R. This restricted version can be stated as

$$\phi\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi\left(Re^{it}\right)}{R^2 + r^2 - 2rR\cos(t-\theta)} dt \tag{2}$$

where ϕ is harmonic on a domain containing the disk $|z| \le R$ and $0 \le r \le R$.

Although the integral in 1 is difficult (or impossible) to do in the general case, there are some special cases where we can make progress.

Example 1. Find

$$u\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{1}{R^2 + r^2 - 2rR\cos(t-\theta)} dt \tag{3}$$

In this case, the function $U\left(Re^{it}\right) = 1$ over the entire circle C_R. From the maximum modulus principle (and the corresponding minimum modulus principle) we know that an analytic function achieves both its maximum and minimum values on the boundary of the domain, so both these moduli must be 1. Hence $u\left(re^{i\theta}\right)$ must be 1 for all points inside C_R as well, that is, it's a constant function. Thus the integral is

$$\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{1}{R^2+r^2-2rR\cos(t-\theta)}dt=1 \tag{4}$$

for all values of R, r and θ, provided $0\le r\le R$.

Incidentally, if you're interested, Maple is able to do the indefinite integral, with the result

$$\int\frac{1}{R^2+r^2-2rR\cos(t-\theta)}dt=-\frac{1}{R^2-r^2}2\arctan\left(\frac{\tan\left(-\frac{t}{2}+\frac{\theta}{2}\right)(R+r)}{R-r}\right) \tag{5}$$

Example 2. Harmonic functions have a number of applications in physics, including the temperature distribution in a 2-d surface. Consider the case of $C_1:|z|=1$ where the temperatures are fixed at $T=0$ in the first quadrant, $T=2$ in the second quadrant, $T=4$ in the third quadrant and $T=6$ in the fourth quadrant. We can use 1 to find the temperature at the centre of the disk.

At the centre, $r=0$, so 1 becomes

$$u(0)=\frac{1}{2\pi}\left[\int_0^{\pi/2}0\,dt+\int_{\pi/2}^{\pi}2\,dt+\int_{\pi}^{3\pi/2}4\,dt+\int_{3\pi/2}^{2\pi}6\,dt\right] \tag{6}$$

$$=\frac{1}{2\pi}(0+\pi+2\pi+3\pi) \tag{7}$$

$$=3 \tag{8}$$

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