POLES OF RATIONAL FUNCTIONS

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If we are given a rational function $R_{m,n}(z)$ of two polynomials where the numerator has degree m and the denominator has degree n, then we can write

$$R_{m,n}(z)=\frac{a_0+a_1z+a_2z^2+\ldots+a_mz^m}{b_n(z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\ldots(z-\zeta_r)^{d_r}} \tag{1}$$

where $\sum_{k=1}^{r}d_k=n$. That is, the sum of the exponents in the denominator equals the degree n. The ζ_ks are the roots of the denominator.

If none of the roots of the numerator are equal to any of the roots in denominator (that is, all common factors have been cancelled), then $R_{m,n}$ is said to have a pole of multiplicity d_k at the value $z=\zeta_k$. Poles will become important in the theory of contour integration, but for now we'll have a look at some examples of poles.

If $R_{m,n}$ is presented in the form 1, where the denominator is already factored, we can just read off the poles by inspection. We do need to ensure that any common factors are cancelled, however.

Example 1. $R_{2,3}=\frac{z^2+4}{(z-2)(z-3)^2}$. In this case, the numerator factors into

$$z^2+4=(z+2i)(z-2i) \tag{2}$$

so there are no common factors, and we have a pole of multiplicity 1 at $z=2$ and of multiplicity 2 at $z=3$.

Example 2. $R_{2,5}=\frac{3z^2+1}{z^3\left(z^2+2iz+1\right)}$. The denominator factors (using the quadratic formula) into

$$z^3\left(z^2+2iz+1\right)=z^3\left(z+\left(1-\sqrt{2}\right)i\right)\left(z+\left(1+\sqrt{2}\right)i\right) \tag{3}$$

The numerator is

$$3z^2+1=3\left(z-\frac{\sqrt{3}}{3}i\right)\left(z+\frac{\sqrt{3}}{3}i\right) \tag{4}$$

so again we have no common factors. $R_{2,5}$ has a pole of multiplicity 3 at $z=0$, and poles of multiplicity 1 at $z=-\left(1-\sqrt{2}\right)i$ and $z=-\left(1+\sqrt{2}i\right)$.

Example 3. $R_{3,6}=\left(\frac{2z+3}{z^2+4z+4}\right)^3$. The denominator factors so we have

$$R_{3,6}=\frac{(2z+3)^3}{(z+2)^6} \tag{5}$$

This has a pole of multiplicity 6 at $z=-2$.

Example 4. $R_{1,2}=\frac{2z}{z^2+3z+2}+\frac{2}{z+1}$. We have

$$R_{1,2}=\frac{2z}{(z+1)(z+2)}+\frac{2}{z+1} \tag{6}$$

$$=\frac{2z+2z+4}{(z+1)(z+2)} \tag{7}$$

$$=\frac{4(z+1)}{(z+1)(z+2)} \tag{8}$$

$$=\frac{4}{z+2} \tag{9}$$

Thus we had to cancel a common factor, and we find that $R_{1,2}$ has a single pole of multiplicity 1 at $z=-2$.

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