POWER SERIES IN INTEGRALS

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Integrands can be expanded in power series, as in the examples below.

Example 1. Let g be continuous on the real interval $[-1,2]$ and define $F(z)$ as

$$F(z)=\int_{-1}^{2}g(t)\sin(zt)\,dt \tag{1}$$

We can expand the sine as

$$\sin(zt)=\sum_{k=0}^{\infty}\frac{(-1)^{k}(zt)^{2k+1}}{(2k+1)!} \tag{2}$$

This gives

$$F(z)=\sum_{k=0}^{\infty}\left[\frac{(-1)^{k}}{(2k+1)!}\int_{-1}^{2}g(t)\,t^{2k+1}dt\right]z^{2k+1} \tag{3}$$

The bracketed term depends only on k (not on z), so this is a power series for $F(z)$ in the form

$$F(z)=\sum_{j=0}^{\infty}a_{j}z^{j} \tag{4}$$

where all even-numbered coefficients are zero. This series for $\sin(zt)$ 2 converges uniformly for all zt and the function $g(t)$ is continuous and hence, bounded, in the interval $[-1,2]$ so termwise multiplication of the sine series by $g(t)$ preserves the convergence. Therefore the series 3 converges uniformly for all z and thus $F(z)$ is entire.

We can take its derivative term by term to get

$$F'(z) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k(2k+1)}{(2k+1)!}\int_{-1}^{2} g(t)\,t^{2k+1}dt\right]z^{2k} \tag{5}$$

$$= \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k)!}\int_{-1}^{2} g(t)\,t^{2k+1}dt\right]z^{2k} \tag{6}$$

$$= \sum_{k=0}^{\infty}\int_{-1}^{2} tg(t)\left[\frac{(-1)^k}{(2k)!}(zt)^{2k}\right]dt \tag{7}$$

$$= \sum_{k=0}^{\infty}\int_{-1}^{2} tg(t)\cos(zt)\,dt \tag{8}$$

Example 2. Show that the forms 1 and 3 give the same result for

$$g(t) = t \tag{9}$$

Admittedly, I used Maple to do the calculations since the algebra gets messy, but we have, from 1

$$F(z) = \int_{-1}^{2} t\sin(zt)\,dt \tag{10}$$

$$= \left.\frac{\sin(zt) - tz\cos(zt)}{z^2}\right|_{t=-1}^{t=2} \tag{11}$$

$$= \frac{-2\cos(2z)\,z - \cos(z)\,z + \sin(2z) + \sin(z)}{z^2} \tag{12}$$

For the series, we have

$$F(z) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k+1)!}\int_{-1}^{2} t^{2k+2}dt\right]z^{2k+1} \tag{13}$$

The integral within the sum is

$$\int_{-1}^{2} t^{2k+2}dt = \frac{1+2^{2k+3}}{2k+3} \tag{14}$$

Thus we have the sum, using Maple

$$F(z) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k+1)!}\frac{1+2^{2k+3}}{2k+3}\right]z^{2k+1} \tag{15}$$

$$= \frac{-\cos(z)\,z + \sin(z)}{z^2} + \frac{-2\cos(2z)\,z + \sin(2z)}{z^2} \tag{16}$$

Comparison with 12 shows the results are the same.

Example 3. Let $g(t)$ be continuous and bounded on the real interval $[0,1]$ and define

$$H(z)=\int_0^1 \frac{g(t)}{1-zt^2}dt \tag{17}$$

The integrand can be expanded as

$$\frac{1}{1-zt^2}=\sum_{k=0}^{\infty}\left(zt^2\right)^k \tag{18}$$

which is valid for $\left|zt^2\right|<1$. Since $t\in[0,1]$, it is valid for $|z|<1$. Putting the series into the integral we get

$$H(z)=\sum_{k=0}^{\infty}\left[\int_0^1 g(t)\,t^{2k}dt\right]z^k \tag{19}$$

Since the series $\sum_{k=0}^{\infty} z^k$ converges for $|z|<1$, multiplying termwise by the bracketed term (which is bounded since $g(t)$ is bounded, and is independent of z) preserves the convergence, so $H(z)$ is analytic for $|z|<1$.

Example 4. Verify 17 and 19 give the same result when

$$g(t)=t \tag{20}$$

From 17 we have

$$H(z)=\int_0^1 \frac{t}{1-zt^2}dt \tag{21}$$

$$=-\left.\frac{\operatorname{Log}\left(1-z\,t^2\right)}{2z}\right|_{t=0}^{t=1} \tag{22}$$

$$=-\frac{\operatorname{Log}\ (1-z)}{2z} \tag{23}$$

From 19 we have for the integral

$$\int_0^1 g(t)\,t^{2k}dt=\int_0^1 t^{2k+1}dt \tag{24}$$

$$=\frac{1}{2(k+1)} \tag{25}$$

so the series is

$$H(z) = \sum_{k=0}^{\infty} \frac{z^k}{2(k+1)} \tag{26}$$

The series for Log $(1-z)$ about $z=0$ for $|z|<1$ is

$$\text{Log }(1-z) = -\sum_{k=0}^{\infty} \frac{z^k}{k} \tag{27}$$

so the series 26 is

$$H(z) = \sum_{k=0}^{\infty} \frac{z^k}{2(k+1)} = -\frac{\text{Log }(1-z)}{2z} \tag{28}$$

which agrees with 23.