POWER SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Post date: 19 February 2025.

A power series can be used to find solutions of ordinary differential equations. A power series (or any Taylor series) can be differentiated or integrated term by term (proof is in Saff and Snider's book, section 5.3), so is a valid solution of a differential equation.

Example 1. Find the power series solution of

$$\frac{df}{dz}=f \tag{1}$$

with initial condition $f(0)=1$. Readers will probably know that the solution is $f(z)=e^{z}$, but we can illustrate using power series. We propose a solution

$$f(z)=\sum_{k=0}^{\infty}a_{k}z^{k} \tag{2}$$

Substituting into 1 we have

$$\frac{df}{dz}=\sum_{k=0}^{\infty}a_{k}kz^{k-1}=f=\sum_{k=0}^{\infty}a_{k}z^{k} \tag{3}$$

The usual method of solving involves collecting terms with common powers of z. To do this, we rewrite the sum on the LHS as

$$\sum_{k=0}^{\infty}a_{k}kz^{k-1}=\sum_{k=0}^{\infty}a_{k+1}(k+1)z^{k} \tag{4}$$

Combining with 3 we have

$$\sum_{k=0}^{\infty}\left(a_{k+1}(k+1)-a_{k}\right)z^{k}=0 \tag{5}$$

In order for this to be satisfied for all z, every coefficient must be zero, so we have

$$a_{k+1}(k+1) - a_k = 0 \tag{6}$$

$$a_{k+1} = \frac{a_k}{k+1} \tag{7}$$

Imposing the initial condition $f(0) = 1$ gives us

$$a_0 = 1 \tag{8}$$

so we get the sequence of coefficients

$$1, 1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!} \ldots, \frac{1}{k!} \tag{9}$$

Thus the solution is

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{10}$$

which is the series representation of e^z, as expected.

Example 2. Power series solution of

$$f'' - zf' - f = 0 \tag{11}$$

with

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \end{aligned} \tag{12}$$

Using 2 we have

$$\sum_{k=0}^{\infty} a_k k(k-1) z^{k-2} - \sum_{k=0}^{\infty} a_k (k+1) z^k = 0 \tag{13}$$

Rewriting the first sum gives

$$\sum_{k=0}^{\infty} \left[a_{k+2}(k+2)(k+1) - a_k(k+1)\right] z^k = 0 \tag{14}$$

We get the recursion relation

$$a_{k+2} = \frac{a_k}{(k+2)} \tag{15}$$

The initial conditions give us

$$f(0) = a_0 = 1 \tag{16}$$

$$f'(0) = a_1 = 0 \tag{17}$$

The sequence is

$$a_k = \left[1, 0, \frac{1}{2}, 0, \frac{1}{2\times 4}, 0, \frac{1}{2\times 4\times 6}, \ldots, \frac{1}{2^{k/2}(k/2)!}\right] \tag{18}$$

This is the series

$$e^{z^2/2} = 1 + \frac{z^2}{2} + \frac{z^4}{2^2 2!} + \frac{z^6}{2^3 3!} + \ldots = \sum_{k=0}^{\infty} \frac{\left(z^2/2\right)^k}{k!} \tag{19}$$

Example 3. Power series solution of

$$f'' + 4f = 0 \tag{20}$$

with initial conditions

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \end{aligned} \tag{21}$$

We have

$$\sum_{k=0}^{\infty} a_k k(k-1) z^{k-2} + 4\sum_{k=0}^{\infty} a_k z^k = 0 \tag{22}$$

$$\sum_{k=0}^{\infty} \left[a_{k+2}(k+2)(k+1) + 4a_k\right] = 0 \tag{23}$$

The recursion relation is

$$a_{k+2} = -\frac{4}{(k+1)(k+2)} a_k \tag{24}$$

The initial conditions give

$$\begin{aligned} f(0) &= a_0 = 1 \\ f'(0) &= a_1 = 1 \end{aligned} \tag{25}$$

We get two interlinked series. The even coefficients are (with a bit of help from Maple):

$$[a_0, a_2, a_4, \ldots] = \left[1, -2, \frac{2}{3}, -\frac{4}{45}, \frac{2}{315}, \ldots\right] \tag{26}$$

This is the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2z)^{2k}}{(2k)!}=\cos 2z \tag{27}$$

The other series is the odd coefficients:

$$[a_1,a_3,a_5,\ldots]=\left[1,-\frac{2}{3},\frac{2}{15},-\frac{4}{315},\ldots\right] \tag{28}$$

This is the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2z)^{2k+1}}{2(2k+1)!}=\frac{\sin(2z)}{2} \tag{29}$$

Thus the solution is

$$f(z)=\frac{\sin(2z)}{2}+\cos(2z) \tag{30}$$

Example 4. Power series solution of

$$\left(1-z^2\right)f''-6zf'-4f=0 \tag{31}$$

with initial conditions

$$\begin{aligned} f(0)&=1\\ f'(0)&=0 \end{aligned} \tag{32}$$

We have

$$\left(1-z^2\right)\sum_{k=0}^{\infty}a_k k(k-1)z^{k-2}-6z\sum_{k=0}^{\infty}a_k k z^{k-1}-4\sum_{k=0}^{\infty}a_k z^k=0 \tag{33}$$

Multiplying through we have

$$\sum_{k=0}^{\infty}a_k k(k-1)z^{k-2}-\sum_{k=0}^{\infty}[k(k-1)+6k+4]a_k z^k=0 \tag{34}$$

Rearranging the first sum we have

$$\sum_{k=0}^{\infty}[a_{k+2}(k+2)(k+1)-[k(k-1)+6k+4]a_k]z^k=0 \tag{35}$$

The recursion relation is

$$a_{k+2} = \frac{k(k-1)+6k+4}{(k+2)(k+1)} a_k \tag{36}$$

$$= \frac{(k+4)(k+1)}{(k+2)(k+1)} a_k \tag{37}$$

$$= \frac{k+4}{k+2} a_k \tag{38}$$

The initial conditions give

$$\begin{aligned} f(0) &= 1 = a_0 \\ f'(0) &= 0 = a_1 \end{aligned} \tag{39}$$

Thus the series consists of even numbered coefficients. The sequence of coefficients turns out to be the positive integers. That is

$$[a_0, a_2, a_4, \ldots] = [1, 2, 3, 4, \ldots] \tag{40}$$

Thus the series is

$$f(z) = \sum_{k=0}^{\infty} (k+1) z^{2k} \tag{41}$$

For $|z| < 1$ this has the closed form (using Maple):

$$f(z) = \frac{1}{(z^2-1)^2} \tag{42}$$